The Laplace Residual Power Series Method for Solving Fractional Fifth-Order Korteweg–de Vries Equations

Ahmed Hagag[1,*], Zuhur Alqahtani[2] and Areej Almuneef[2]

[1] Department of Basic Science, Faculty of Engineering, Sinai University—Kantara Branch, Ismaili, 41636, Egypt

[2] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh, 11671, Saudi Arabia

INFORMATION

Keywords:
Fractional fifth-order Korteweg–de Vries (KdV) equations
fractional Sawada–Kotera equation
fractional Caudrey–Dodd–Gibbon equation
fractional Kaup–Kuperschmidt equation
Laplace residual power series method

DOI: 10.23967/j.rimni.2025.10.62089

SCIPEDIA
Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería

The Laplace Residual Power Series Method for Solving Fractional Fifth-Order Korteweg–de Vries Equations

Ahmed Hagag[1,*], **Zuhur Alqahtani**[2] **and Areej Almuneef**[2]

[1]Department of Basic Science, Faculty of Engineering, Sinai University—Kantara Branch, Ismaili, 41636, Egypt

[2]Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh, 11671, Saudi Arabia

ABSTRACT

This study explores a modern analytical approach for solving the fractional fifth-order Korteweg–de Vries (KdV) equations, which describe intricate wave phenomena influenced by nonlinearity, dispersion, and memory effects. Specifically, the Laplace residual power series method (LRPSM) is utilized to obtain accurate approximate analytical solutions for three fundamental fractional equations: the fractional Sawada–Kotera (SK) equation, the fractional Caudrey–Dodd–Gibbon (CDG) equation, and the fractional Kaup–Kuperschmidt (KK) equation. These equations represent special cases of the broader fractional fifth-order KdV equation. The novelty of this study lies in the application of LRPSM, which addresses the limitations of traditional methods by combining analytical precision with computational efficiency. The method successfully captures fractional dynamics, including soliton-like behaviors and memory effects, demonstrating its capability to model wave attenuation and smoothness influenced by fractional orders. The numerical results demonstrate that this method achieves minimal error margins, validating its robustness and precision in solving nonlinear fractional systems. Numerical examples validate the efficiency and robustness of this method, achieving high accuracy in solving nonlinear fractional systems. The results establish LRPSM as a versatile and reliable tool for solving fractional differential equations, paving the way for advancements in modern wave theory and applications across disciplines such as plasma physics, fluid mechanics, and nonlinear optics.

OPEN ACCESS

Received: 12/10/2024

Accepted: 21/01/2025

Published: 14/07/2025

DOI
10.23967/j.rimni.2025.10.62089

Keywords:
Fractional fifth-order Korteweg–de Vries (KdV) equations
fractional Sawada–Kotera equation
fractional Caudrey–Dodd–Gibbon equation
fractional Kaup–Kuperschmidt equation
Laplace residual power series method

1 Introduction

Fractional calculus extends classical integer-order calculus to fractional domains. While the concept dates back to the 17th Century, its practical applications have only recently gained widespread recognition across scientific and engineering disciplines [1,2]. Unlike integer-order calculus, fractional derivatives describe systems influenced by their past states (memory effects) and distant interactions,

*Correspondence: Ahmed Hagag (ahmed.shehata@su.edu.eg).

making them useful for modeling complex physical processes. For instance, in viscoelastic materials, stress depends not only on the current strain rate but also on its history, a feature naturally described by fractional derivatives [3,4].

A fundamental distinction between integer-order and fractional-order calculus lies in the interpretation of derivatives. While integer-order derivatives provide rates of change at an instant, fractional derivatives capture rates of change over an interval, influenced by prior states of the system. This nonlocal nature allows fractional models to bridge the gap between microscopic and macroscopic phenomena, yielding superior descriptions of complex physical processes [5]. The fractional derivative order\alpha modulates the extent of memory effects in the system, with lower values leading to increased dispersive effects and smoother wave profiles. In fluid mechanics, for example, fractional models have been employed to analyze anomalous diffusion, which occurs when particle transport deviates from classical Brownian motion due to heterogeneities or external constraints [6,7]. Similarly, fractional calculus has been applied in control theory to design more robust systems, in finance to model market dynamics, and in biophysics to understand subdiffusive behaviors in cellular processes [8–11].

In mathematical physics, fractional partial differential equations (FPDEs) have emerged as critical models for describing nonlinear phenomena. These equations incorporate fractional derivatives to account for memory and hereditary effects, as well as long-range interactions, enabling more accurate descriptions of real-world systems [12]. Among FPDEs, fractional versions of the Korteweg–de Vries (KdV) equation have gained significant attention. The KdV equation, originally introduced to describe shallow water waves, has since been generalized to higher-order and fractional forms, broadening its applicability to plasma physics, nonlinear optics, and other fields [13,14]. The fractional fifth-order KdV equation, a prominent example of such generalizations, is expressed as

$$D_t^\alpha u + au^2u_x + bu_xu_{xx} + cuu_{xxx} + du_{xxxxx} = 0, 0 < \alpha \leq 1, \tag{1}$$

represents a significant advancement in the work of nonlinear waves, extending the classical KdV equation by incorporating higher-order dispersion and nonlocal effects. Here, D_t^α denotes the fractional derivative of order α, and the coefficients a, b, c, d determine the nonlinear and dispersive properties of the system.

Each term in the equation plays a distinct role in characterizing wave behavior. The nonlinear term au^2u_x describes the self-interaction of waves, leading to steepening or shock formation. The dispersive terms bu_xu_{xx}, cuu_{xxx} and du_{xxxxx} introduce higher-order effects, allowing the model to account for phenomena such as wave splitting or stabilization of sharp gradients. The fractional derivative D_t^α generalizes the time evolution of the system, capturing memory effects and broadening the range of applicable physical scenarios [15,16]. This equation finds applications in plasma physics, describing the dynamics of ion-acoustic waves, and in fluid mechanics, where it models shallow water wave propagation in heterogeneous environments [17,18].

A distinctive feature of the fractional fifth-order KdV equation is its capacity to generalize into specialized equations by assigning specific values to the coefficients a,b,c,d. Among these, the fractional Sawada-Kotera (SK) equation,

$$D_t^\alpha u + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx} = 0, \tag{2}$$

is well-known for its applications in describing dispersive waves and energy transport in nonlinear systems [19,20]. This equation models phenomena such as solitary waves in shallow water and nonlinear optical systems. The fractional Caudrey-Dodd-Gibbon (CDG) equation,

$$D_t^{\alpha} u + 180u^2 u_x + 30u_x u_{xx} + 30uu_{xxx} + u_{xxxxx} = 0, \tag{3}$$

is particularly valuable for understanding systems dominated by strong nonlinearities and weak dispersion, such as plasma waves and certain fluid flows [21]. The fractional Kaup-Kuperschmidt (KK) equation,

$$D_t^{\alpha} u + 20u^2 u_x + 25u_x u_{xx} + 10uu_{xxx} + u_{xxxxx} = 0, \tag{4}$$

is an integrable system that plays a significant role in wave theory and quantum mechanics. Its solutions provide insights into the dynamics of nonlinear wave propagation in nonhomogeneous media [22,23].

The study of these equations reveals a rich variety of solutions, including bright and dark solitons, singular solutions, and periodic wave structures. Such solutions have practical implications across numerous domains. For instance, soliton solutions are critical in telecommunications for signal transmission in optical fibers and are also observed in Bose-Einstein condensates, where they describe localized wave packets [24]. Understanding the dynamics of these solutions provides a deeper insight into the behavior of nonlinear systems, facilitating advancements in both theory and applications [25].

To solve fractional differential equations like the fifth-order KdV equation and its derivatives, robust analytical and numerical methods are essential. Traditional approaches, such as the Adomian decomposition method and variational techniques, often face challenges with convergence and accuracy when applied to fractional equations [26]. In this study, the LRPSM is employed to address these limitations. LRPSM combines the analytical precision of power series solutions with the computational efficiency of Laplace transforms, enabling effective handling of nonlinear and fractional terms [27,28].

One of the key advantages of LRPSM is its adaptability to different fractional orders and boundary conditions. By iteratively refining solutions, the method ensures rapid convergence and high accuracy, even for complex equations. Additionally, LRPSM preserves the nonlocal nature of fractional derivatives, allowing for a faithful representation of memory effects and long-range interactions [29]. These features make LRPSM a powerful tool for exploring the rich dynamics of fractional systems and deriving solutions with practical significance [30].

The novelty of this work lies in the application of LRPSM to the fractional fifth-order KdV equation and its derivatives, specifically the SK, CDG, and KK equations. By deriving explicit solutions and analyzing their behavior, this study contributes to the understanding of nonlinear wave dynamics and demonstrates the potential of LRPSM in tackling complex FPDEs. Moreover, the solutions obtained here provide a foundation for future research into fractional systems and their applications in physics, engineering, and beyond [31,32].

This paper is organized into six sections. Section 1 introduces the fractional fifth-order Korteweg–de Vries equation and its significance in describing nonlinear wave phenomena influenced by nonlinearity, dispersion, and memory effects. Section 2 provides the mathematical preliminaries of fractional calculus, including key definitions and fundamental concepts such as fractional derivatives and integrals. Section 3 describes the LRPSM in detail, outlining its theoretical foundation and computational procedure for solving nonlinear fractional equations. Section 4 focuses on numerical examples, applying LRPSM to the fractional SK equation, the fractional CDG equation, and the fractional KK equation, with results validated through graphical representations and error analysis. Section 5 discusses the numerical results and their physical interpretations, emphasizing the accuracy and applicability of LRPSM in modeling fractional systems. The study concludes with a summary of the main conclusions for LRPSM in Section 6.

2 Preliminaries

The basic meaning of FC and Laplace transform (LT) is provided here [33,34].

Definition 1

The Riemann-Liouville integral of a function $u(t) \in C_\delta(\delta \geq -1)$ having fractional order $(\alpha > 0)$ is presented as follows:

$$J^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau. \tag{5}$$

Definition 2

The Caputo fractional order derivative of $u \in C_{-1}^n$ is presented as

$$D_t^\alpha = \begin{cases} \dfrac{d^n u(t)}{dt^n}, (\sigma = n \in N), \\ \dfrac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{(n-\alpha-1)} u^{(n)}(\tau)\, d\tau, ((n-1) < \sigma < n, n \in N). \end{cases} \tag{6}$$

3 Construction of Fractional TAM

The LRPSM generates a sequence of improved approximations to a collection of problems. The iterative method yields an approximate solution that converges to the precise answer when the proper sequence converges at a set of starting approximations.

First, we convert (1) using the Laplace transform as

$$\mathcal{L}\left[D_t^\alpha u\right] + a\mathcal{L}\left[u^2 u_x\right] + b\mathcal{L}\left[u_x u_{xx}\right] + c\mathcal{L}\left[u u_{xxx}\right] + d\mathcal{L}\left[u_{xxxxx}\right] = 0. \tag{7}$$

Using the initial condition and based on the knowledge that $\mathcal{L}\left[D_t^\alpha u(x,t)\right] = s^\alpha \mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0)$, we rewrite (7) as

$$\begin{aligned} U(x,s) = {} & \frac{f_0}{s} - \frac{a}{s^\alpha}\mathcal{L}\left[\left(\mathcal{L}^{-1}[U(x,s)]\right)^2\left(\mathcal{L}^{-1}\left[U_x(x,s)\right]\right)\right] - \frac{b}{s^\alpha}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[U_x(x,s)\right]\right)\left(\mathcal{L}^{-1}\left[U_{xx}(x,s)\right]\right)\right] \\ & -\frac{c}{s^\alpha}\mathcal{L}\left[\mathcal{L}^{-1}[U(x,s)]\left(\mathcal{L}^{-1}\left[U_{xxx}(x,s)\right]\right)\right] - \frac{d}{s^\alpha}U_{xxxxx}(x,s), \end{aligned} \tag{8}$$

where $U(x,s) = \mathcal{L}[u(x,t)]$.

Second, we construct the expanded form of the transformed function $U(x,s)$

$$U(x,s) = \sum_{n=0}^{\infty} \frac{f_n(x)}{s^{n\alpha+1}}. \tag{9}$$

The k-th-truncated series of Eq. (9) is represented by

$$U_k(x,s) = \sum_{n=0}^{k} \frac{f_n(x)}{s^{n\alpha+1}} = \frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}. \tag{10}$$

The Laplace residual function to Eq. (9) is defined as follows:

$$\begin{aligned}\mathcal{L}\mathrm{Res}(x,s) \;=\; & U(x,s)-\frac{f_0}{s}+\frac{a}{s^\alpha}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[U\left(x,s\right)\right]\right)^2\left(\mathcal{L}^{-1}\left[U_x\left(x,s\right)\right]\right)\right]+\frac{b}{s^\alpha}\mathcal{L}[\left(\mathcal{L}^{-1}\left[U_x(x,s)\right]\right) \\ & \left(\mathcal{L}^{-1}\left[U_{xx}(x,s)\right]\right)]+\frac{c}{s^\alpha}\mathcal{L}\left[\mathcal{L}^{-1}[U(x,s)]\left(\mathcal{L}^{-1}\left[U_{xxx}(x,s)\right]\right)\right]+\frac{d}{s^\alpha}U_{xxxxx}(x,s).\end{aligned} \tag{11}$$

The k-th-truncated series of Eq. (11) is represented as

$$\begin{aligned}\mathcal{L}\mathrm{Res}_k(x,s) \;=\; & U_k(x,s)-\frac{f_0}{s}+\frac{a}{s^\alpha}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[U_k\left(x,s\right)\right]\right)^2\left(\mathcal{L}^{-1}\left[\left(U_x\right)_k\left(x,s\right)\right]\right)\right]+\frac{b}{s^\alpha}\mathcal{L}[\left(\mathcal{L}^{-1}\left[\left(U_x\right)_k\left(x,s\right)\right]\right) \\ & \left(\mathcal{L}^{-1}\left[\left(U_{xx}\right)_k\left(x,s\right)\right]\right)]+\frac{c}{s^\alpha}\mathcal{L}\left[\mathcal{L}^{-1}\left[U_k(x,s)\right]\left(\mathcal{L}^{-1}\left[\left(U_{xxx}\right)_k\left(x,s\right)\right]\right)\right]+\frac{d}{s^\alpha}\left(U_{xxxxx}\right)_k(x,s).\end{aligned} \tag{12}$$

Third, we highlight a few facts by extending several qualities that come up in the normal residual power series technique

- $\mathcal{L}\mathrm{Res}(x,s)=0$ and $\lim_{k\to\infty}\mathcal{L}\mathrm{Res}_k(x,s)=\mathcal{L}\mathrm{Res}(x,s)$ for every $s>0$.
- $\lim_{s\to\infty}\mathcal{L}\mathrm{Res}(x,s)=0\Rightarrow\lim_{s\to\infty}s\mathcal{L}\mathrm{Res}_k(x,s)=0$.
- $\lim_{s\to\infty}s^{k\alpha+1}\mathcal{L}\mathrm{Res}(x,s)=\lim_{s\to\infty}s^{k\alpha+1}\mathcal{L}\mathrm{Res}_k(x,s)=0, 0<\alpha\le 1, k=1,2,3,\cdots$.

In order to obtain the coefficient functions $f_n(x)$, we repeatedly solve the following system:

$$\lim_{s\to\infty}\left(s^{k\alpha+1}\mathcal{L}\mathrm{Res}_k\left(x,s\right)\right)=0, 0<\alpha\le 1, k=1,2,3,\cdots$$

Finally, the analytical solution $u_k(x,t)$ is obtained by applying the Laplace inverse to $U_k(x,s)$.

4 Applications by TAM

This section demonstrates how the iterative method LRPSM will be used to solve the fractional fifth-order Korteweg–de Vries equations.

4.1 Example 1

To express the fractional Sawada-Kotera equation, we consider the following format [35]:

$$D_t^\sigma u(x,t)+45u^2u_x+15u_xu_{xx}+15uu_{xxx}+u_{5x}=0, \tag{13}$$

where $a=45, b=c=15$ and $d=1$. With initial conditions

$$\begin{aligned}u(x,0)&=\frac{20k^3\pm\sqrt{5}\sqrt{kh+4h^6}}{15k}-2k^2\tanh^2\left(kx+w\right),\\ u^*(x,0)&=\frac{\pm\sqrt{5}\sqrt{kh+4k^6}-10k^3}{15h}+2k^2\mathrm{sech}^2\left(kx+w\right).\end{aligned} \tag{14}$$

To begin the LRPS method stages, we apply the Laplace transform on (13) and utilize the initial condition (14) to obtain

$$\begin{aligned} U(x,s) &= \frac{f_0}{s} - \frac{45}{s^\alpha}\mathcal{L}\left[\left(\mathcal{L}^{-1}[U(x,s)]\right)^2\left(\mathcal{L}^{-1}[U_x(x,s)]\right)\right] - \frac{15}{s^\alpha}\mathcal{L}\left[\left(\mathcal{L}^{-1}[U_x(x,s)]\right)\left(\mathcal{L}^{-1}[U_{xx}(x,s)]\right)\right] \\ &\quad - \frac{15}{s^\alpha}\mathcal{L}\left[\mathcal{L}^{-1}[U(x,s)]\left(\mathcal{L}^{-1}[U_{xxx}(x,s)]\right)\right] - \frac{1}{s^\alpha}U_{xxxxx}(x,s). \end{aligned} \tag{15}$$

As a result, we can write the k-th-truncated series of Eq. (15) as

$$U_k(x,s) = \frac{f_0}{s} + \sum\nolimits_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}, \tag{16}$$

and the k-th-Laplace residual function is

$$\begin{aligned} \mathcal{L}\mathrm{Res}_k(x,s) &= U_k(x,s) - \frac{f_0}{s} + \frac{45}{s^\alpha}\mathcal{L}\left[\left(\mathcal{L}^{-1}[U_k(x,s)]\right)^2\left(\mathcal{L}^{-1}[(U_x)_k(x,s)]\right)\right] + \frac{15}{s^\alpha}\mathcal{L}[\left(\mathcal{L}^{-1}[(U_x)_k(x,s)]\right) \\ &\quad \left(\mathcal{L}^{-1}[(U_{xx})_k(x,s)]\right)] + \frac{15}{s^\alpha}\mathcal{L}\left[\mathcal{L}^{-1}[U_k(x,s)]\left(\mathcal{L}^{-1}[(U_{xxx})_k(x,s)]\right)\right] + \frac{1}{s^\alpha}(U_{xxxxx})_k(x,s). \end{aligned} \tag{17}$$

Now, we replace the series (16) with Eq. (17), multiply the resultant equation by $s^{k\alpha+1}$, and then solve the relation $\lim_{s\to\infty}\left(s^{k\alpha+1}\mathcal{L}\mathrm{Res}(x,s)\right) = 0, k = 1,2,3,\ldots$ recursively to obtain $f_k(x), k = 1,2,3,\cdots$. The first few components of the sequence $\{f_k(x)\}$ are shown below:

$$\begin{aligned} f_1(x) &= -45u_0^2(u_0)_x - 15(u_0)_x(u_0)_{xx} - 15u_0(u_0)_{xxx} - (u_0)_{xxxxx} \\ &= 4k^2h\tanh(kx+w)\,\mathrm{sech}^2(kx+w), \end{aligned} \tag{18}$$

$$\begin{aligned} f_2(x) &= -45u_0^2(u_1)_x - 90u_1u_0(u_0)_x - 15(u_0)_x(u_1)_{xx} - 15(u_1)_x(u_0)_{xx} - 15u_1(u_0)_{xxx} \\ &\quad -15u_0(u_1)_{xxx} - (u_1)_{xxxxx} \\ &= 4k^2h^2(\cosh(2(kx+w)) - 2)\,\mathrm{sech}^4(kx+w), \end{aligned} \tag{19}$$

$$\begin{aligned} f_3(x) &= -\frac{90u_1u_0\Gamma(2\alpha+1)(u_1)_x}{\Gamma(\alpha+1)^2} - \frac{45u_1^2\Gamma(2\alpha+1)(u_0)_x}{\Gamma(\alpha+1)^2} - 45u_0^2(u_2)_x - 90u_2u_0(u_0)_x \\ &\quad -\frac{15u_1\Gamma(2\alpha+1)(u_1)_{xxx}}{\Gamma(\alpha+1)^2} - \frac{15\Gamma(2\alpha+1)(u_1)_x(u_1)_{xx}}{\Gamma(\alpha+1)^2} - 15(u_2)_x(u_0)_{xx} - 15(u_0)_x(u_2)_{xx} \\ &\quad -15u_2(u_0)_{xxx} - 15u_0(u_2)_{xxx} - (u_2)_{xxxxx}, \end{aligned} \tag{20}$$

$$\begin{aligned} f_3(x) &= \frac{k^2h^2}{2\Gamma(\alpha+1)^2}\Big\{\Big[-25920k^5 - 26h + 288\sqrt{5}k^2\sqrt{k(4k^5+h)} + 3\left(6400k^5 + 64\sqrt{5}k^2\sqrt{k(4k^5+h)} - 11h\right) \\ &\quad \cosh(2(kx+w)) - 6\left(160k^5 + 16\sqrt{5}k^2\sqrt{k(4k^5+h)} + h\right)\cosh(4(kx+w)) + h\cosh(6(kx+w))\Big)\Big] \\ &\quad \times\Gamma(\alpha+1)^2 + 48k^2\Big(\Big(270k^3 - 3\sqrt{5}\sqrt{k(4k^5+h)} - 2\left(\sqrt{5}\sqrt{k(4k^5+h)} + 100k^3\right)\cosh(2(kx+w)) \\ &\quad + \left(\sqrt{5}\sqrt{k(4k^5+h)} + 10k^3\right)\cosh(4(kx+w))\Big)\Gamma(2\alpha+1)\Big)\Big\}\,\mathrm{sech}^8(kx+w)\tanh(kx+w), \end{aligned} \tag{21}$$

and so on. Therefore, the infinite series solution of Eqs. (13) and (14) is $U(x,s) = \frac{f_0}{s} + \frac{f_1}{s^{\alpha+1}} + \frac{f_2}{s^{2\alpha+1}} + \cdots$. Finally, the Laplace inverse of Eq. (15), gives that the k^{th}-approximate solution of our problem is

$$u(x,t) = f_0 + \frac{f_1 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{f_2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots . \tag{22}$$

When $\alpha = 1$, the exact solutions of Eqs. (13) and (14) is

$$u(x,t) = \frac{20k^3 \pm \sqrt{5}\sqrt{kh+4k^6}}{15k} - 2k^2 \tanh^2(kx - ht + w), \tag{23}$$

$$u^*(x,t) = \frac{\pm\sqrt{5}\sqrt{kh+4k^6} - 10k^3}{15k} + 2k^2 \operatorname{sech}^2(kx - ht + w). \tag{24}$$

4.2 Example 2

To express the fractional Caudrey-Dodd-Gibbon equation, we consider the following format [35,36]:

$$D_t^{\alpha} u(x,t) + 180u^2 u_x + 30u_x u_{xx} + 30uu_{xxx} + u_{xxxxx} = 0, \tag{25}$$

where $a = 180, b = c = 30$ and $d = 1$. With initial conditions

$$u(x,0) = \frac{\pm\sqrt{5}\sqrt{kh+4k^6} - 10k^3}{30k} - k^2 \operatorname{csch}^2(kx + w), \tag{26}$$

$$u^*(x,0) = \frac{20k^3 \pm \sqrt{5}\sqrt{kh+4k^6}}{15k} - k^2 \coth^2(kx + w). \tag{27}$$

To begin the LRPS method stages, we apply the Laplace transform on Eq. (25) and utilize the initial condition (26) to obtain

$$\begin{aligned} U(x,s) = & \frac{f_0}{s} - \frac{180}{s^{\alpha}} \mathcal{L}\left[\left(\mathcal{L}^{-1}[U(x,s)]\right)^2 \left(\mathcal{L}^{-1}[U_x(x,s)]\right)\right] - \frac{30}{s^{\alpha}} \mathcal{L}\left[\left(\mathcal{L}^{-1}[U_x(x,s)]\right)\left(\mathcal{L}^{-1}[U_{xx}(x,s)]\right)\right] \\ & - \frac{30}{s^{\alpha}} \mathcal{L}\left[\mathcal{L}^{-1}[U(x,s)]\left(\mathcal{L}^{-1}[U_{xxx}(x,s)]\right)\right] - \frac{1}{s^{\alpha}} U_{xxxxx}(x,s). \end{aligned} \tag{28}$$

As a result, we can write the k-th-truncated series of Eq. (28) as

$$U_k(x,s) = \frac{f_0}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}, \tag{29}$$

and the k-th-Laplace residual function is

$$\begin{aligned} \mathcal{L}Res_k(x,s) = & U_k(x,s) - \frac{f_0}{s} + \frac{180}{s^{\alpha}} \mathcal{L}\left[\left(\mathcal{L}^{-1}[U_k(x,s)]\right)^2 \left(\mathcal{L}^{-1}[(U_x)_k(x,s)]\right)\right] + \frac{30}{s^{\alpha}} \mathcal{L}[(\mathcal{L}^{-1} \\ & (U_x)_k(x,s)])\left(\mathcal{L}^{-1}[(U_{xx})_k(x,s)]\right)] + \frac{30}{s^{\alpha}} \mathcal{L}\left[\mathcal{L}^{-1}[U_k(x,s)]\left(\mathcal{L}^{-1}[(U_{xxx})_k(x,s)]\right)\right] \\ & + \frac{1}{s^{\alpha}} (U_{xxxxx})_k(x,s) \end{aligned} \tag{30}$$

Now, we replace the series (29) with Eq. (30), multiply the resultant equation by $s^{k\alpha+1}$, and then solve the relation $\lim_{s\to\infty}\left(s^{k\alpha+1}\mathcal{L}\mathrm{Res}_k(x,s)\right) = 0, k = 1,2,3,\ldots$ recursively to obtain $f_k(x), k = 1,2,3,\cdots$. The first few components of the sequence $\{f_k(x)\}$ are shown below:

$$\begin{aligned} f_1(x) &= -180u_0^2(u_0)_x - 30(u_0)_x(u_0)_{xx} - 30u_0(u_0)_{xxx} - (u_0)_{xxxxx} \\ &= -8k^2\left(8k^5 - 2\sqrt{5}k^2\sqrt{k(4k^5+h)} + h\right)\coth(kx+w)\mathrm{csch}^2(kx+w) \end{aligned} \tag{31}$$

$$\begin{aligned} f_2(x) &= -360u_0^2(u_1)_x - 180u_1u_0(u_0)_x - 30(u_0)_x(u_1)_{xx} - 30(u_1)_x(u_0)_{xx} - 30u_1(u_0)_{xxx} \\ &\quad -30u_0(u_1)_{xxx} - (u_1)_{xxxxx} \\ &= -32\left(8k^6 - 2\sqrt{5}k^3\sqrt{k(4k^5+h)} + kh\right)^2(\cosh(2(kx+w))+2)\mathrm{csch}^4(kx+w), \end{aligned} \tag{32}$$

$$\begin{aligned} f_3(x) &= -360u_2u_0(u_0)_x - \frac{360u_1u_0\Gamma(2\alpha+1)(u_1)_x}{\Gamma(\alpha+1)^2} - 180u_0^2(u_2)_x - \frac{180u_1^2\Gamma(2\alpha+1)(u_0)_x}{\Gamma(\alpha+1)^2} \\ &\quad -30(u_0)_x(u_2)_{xx} - 30(u_2)_x(u_0)_{xx} - 30u_0(u_2)_{xxx} - \frac{30\Gamma(2\alpha+1)(u_1)_x(u_1)_{xx}}{\Gamma(\alpha+1)^2} - 30u_2(u_0)_{xxx} \\ &\quad -\frac{30\Gamma(2\alpha+1)u_1(u_1)_{xxx}}{\Gamma(\alpha+1)^2} - (u_2)_{xxxxx}, \end{aligned} \tag{33}$$

and so on. Therefore, the infinite series solution of Eqs. (25) and (26) is $U(x,s) = \frac{f_0}{s} + \frac{f_1}{s^{\alpha+1}} + \frac{f_2}{s^{2\alpha+1}} + \cdots$. Finally, the Laplace inverse of Eq. (28), gives that the k-th-approximate solution of our problem is

$$u(x,t) = f_0 + \frac{f_1t^\alpha}{\Gamma(1+\alpha)} + \frac{f_2t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots. \tag{34}$$

When $\alpha = 1$, the exact solutions of Eqs. (25) and (26) is

$$\begin{aligned} u(x,t) &= \frac{\pm\sqrt{5}\sqrt{kh+4k^6} - 10k^3}{30k} - k^2\mathrm{csch}^2(kx-ht+w), \\ u^*(x,t) &= \frac{20k^3 \pm \sqrt{5}\sqrt{kh+4k^6}}{15k} - k^2\coth^2(kx-ht+w). \end{aligned} \tag{35}$$

4.3 Example 3

To express the fractional Kaup-Kuperschmidt equation, we consider the following format [36]:

$$D_t^\sigma u(x,t) + 20u^2u_x + 25u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0, \tag{36}$$

where $a = 20$, $b = 25$, $c = 10$ and $d = 1$. With initial conditions

$$u(x,0) = -h^2 + \frac{12s^2e^{hx}}{(1+e^{hx})^2}, \tag{37}$$

$$u^*(x,0) = -\frac{h^2}{8} + \frac{3h^2e^{hx}}{2(1+e^{hx})^2}. \tag{38}$$

Based on the above in the previous two examples, $\{f_k(x)\}$ can be calculated and written directly as follows:

$$f_1(x) = \frac{132h^7e^{hxx}(e^{hx}-1)}{(e^{hxx}+1)^3}, \tag{39}$$

$$f_2(x) = \frac{1452h^{12}e^{hx}\left(-4e^{hx} + e^{2hx} + 1\right)}{\left(e^{hx} + 1\right)^4}, \tag{40}$$

$$f_3(x) = \frac{1452h^{17}e^{hx}\left(e^{hxx} - 1\right)}{\Gamma(\alpha+1)^2\left(e^{hx}+1\right)^9}\left\{\Gamma(\alpha+1)^2(54e^{hx} - 4923e^{2hx} + 10228e^{3hx} - 4923e^{4hx} + 54e^{5hx} + 11e^{6hx} + 11) - 60\Gamma(2\alpha+1)e^{hx}\left(-38e^{hx} + 90e^{2hx} - 38e^{3hx} + e^{4hx} + 1\right)\right\}, \tag{41}$$

and so on. Therefore, the infinite series solution of Eqs. (36) and (37) is $U(x,s) = \frac{f_0}{s} + \frac{f_1}{s^{\alpha+1}} + \frac{f_2}{s^{2\alpha+1}} + \cdots$. Finally, the Laplace inverse gives that the k-th-approximate solution of our problem is

$$u(x,t) = f_0 + \frac{f_1 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{f_2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots \tag{42}$$

When $\alpha = 1$, the exact solutions of Eqs. (36) and (37) is

$$u(x,t) = -h^2 + \frac{12h^2 e^{h\left(x-11k^4 t\right)}}{\left(1 + e^{h\left(x-11k^4 t\right)}\right)^2}, \tag{43}$$

$$u^*(x,t) = -\frac{h^2}{8} + \frac{3h^2 e^{h\left(x-\frac{k^4}{16}t\right)}}{2\left(1 + e^{h\left(x-\frac{k^4}{16}t\right)}\right)^2}. \tag{44}$$

5 Numerical Results and Discussion

The study explores the LRPSM applied to the fractional fifth-order KdV equations and its three notable variants: the fractional SK, CDG, and KK equations. Each example demonstrates the method's efficiency and robustness in deriving approximate analytical solutions while analyzing the effect of fractional orders α on solution accuracy. The results are supported by figures and tables that compare LRPSM solutions with exact solutions.

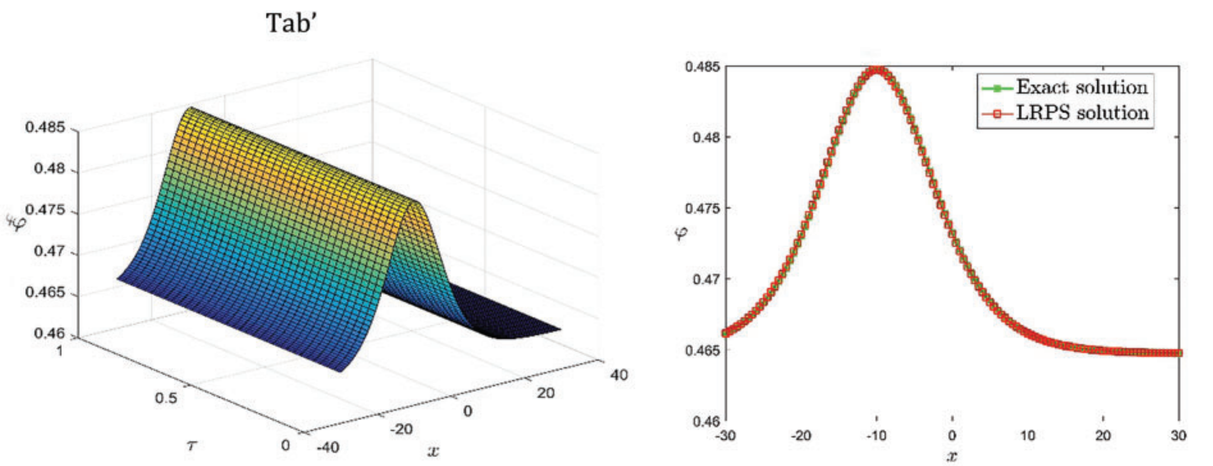
The fractional SK equation uses coefficients $a = 45, b = c = 15$, and $d = 1$, with initial conditions defined using hyperbolic functions, including $\text{sech}^2(kx + w)$ and $\tanh^2(kx + w)$. Fig. 1a shows the approximate solution derived using LRPSM, capturing the soliton-like behavior typical of this model. Fig. 1b overlays the approximate solution with the exact solution, illustrating close agreement for $\alpha = 1$. For lower fractional orders ($\alpha = 0.9, 0.8$), deviations become apparent, indicating increased dispersive effects caused by memory influences introduced by fractional derivatives. Table 1 presents exact and approximate solutions for various values of x, with relative errors computed for each case. At $\alpha = 1$, the errors are negligible across the domain $x = -20$ to $x = 20$. However, for $\alpha = 0.8$, the error increases slightly, especially near sharp gradient changes, demonstrating the impact of fractional dynamics on solution behavior.

The fractional CDG equation involves coefficients $a = 180$, $b = c = 30$, and $d = 1$, with initial conditions incorporating $\text{csch}^2(kx+w)$ and $\coth^2(kx+w)$. This equation is known for modeling systems with pronounced nonlinearities and weaker dispersion.

Fig. 2 depicts the wave behavior under different fractional orders. For $\alpha = 1$, the solution maintains periodicity and sharply localized structures. As α decreases to 0.9 and 0.8, the waveform begins to attenuate, showing smoother profiles due to the enhanced memory effects. Table 2 compares LRPSM-derived solutions with exact values for various x values. The errors remain minimal for

$\alpha = 1.0$ and increase slightly for fractional orders, particularly near steep gradients. This behavior highlights the effect of fractional parameters in modulating energy dissipation and wave propagation dynamics.

The fractional KK equation involves coefficients $a = 20$, $b = 25$, $c = 10$, and $d = 1$, with initial conditions expressed using exponential functions $e^{-h(x-t)}$. This equation is significant for studying integrable systems and wave behavior in non-homogeneous media.

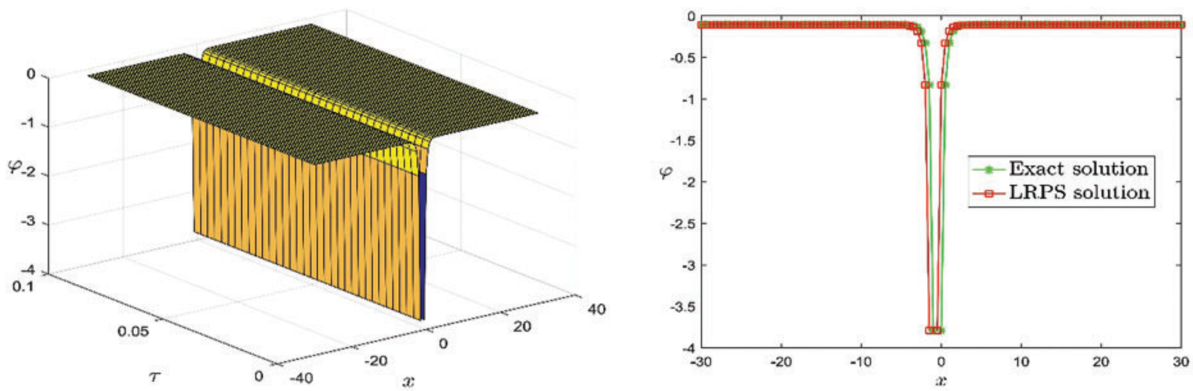


(a) Approximate solution of $u(x,t)$. (b) The approximate solution with the exact solution.

Figure 1: Behaviour of group of 3rd order LRPSM of Eq. (13) at $h = 1$, $k = 0.1$, $w = 1$ where subgraph (a) shows the LRPSM solution of the fractional SK equation and subgraph (b) compares the LRPSM solution with the exact solution for $\alpha = 1$

Table 1: Compare analytical solutions $u(x, t)$ of Eq. (13) obtained by LRPSM with exact for Example 1 at h = 3, k = 1, w = 1

x	Exact	u_{LRPS}	Error	$\alpha = 0.9$	$\alpha = 0.8$
−20	−0.272261	−0.272261	5.42413E − 15	−0.272261	−0.272261
−15	−0.272261	−0.272261	1.27239E − 10	−0.272261	−0.272261
−10	−0.272261	−0.272264	2.80262E − 06	−0.272266	−0.272268
−5	−0.272255	−0.333578	6.13238E − 02	−0.364861	−0.407048
5	−0.252529	−0.269263	1.67339E − 02	−0.268305	−0.267099
10	−0.272260	−0.272261	7.64155E − 07	−0.272261	−0.272261
15	−0.272261	−0.272261	3.46925E − 11	−0.272261	−0.272261
20	−0.272261	−0.272261	1.55431E − 15	−0.272261	−0.272261



(a) Approximate solution of $u(x,t)$. (b) The approximate solution with the exact solution.

Figure 2: Behaviour of group of 3rd order LRPSM of Eq. (25) at $h = 10$, $k = 1$, $w = 1$ where subgraph (a) shows the LRPSM solution of the fractional CDG equation and subgraph (b) compares the LRPSM solution with the exact solution for $\alpha = 1$

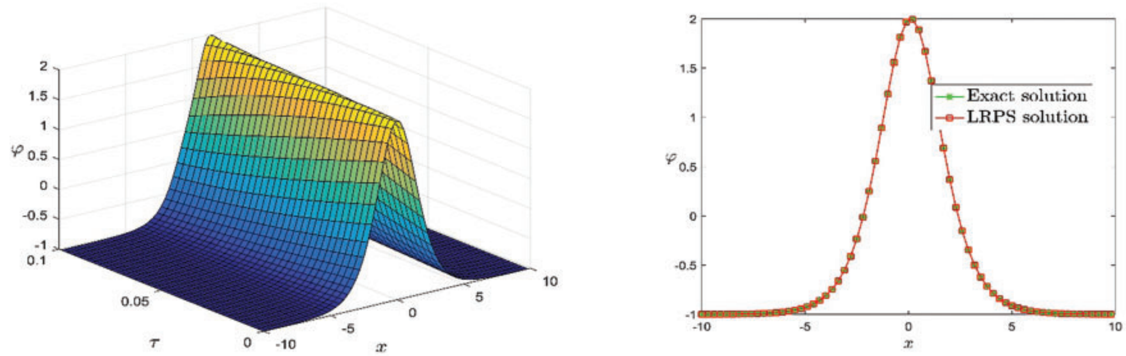
Table 2: Compare analytical solutions $u(x, t)$ of Eq. (25) obtained by LRPSM with exact for Example 2 at h = 5, k = 1, w = 1

x	Exact	u_{LRPS}	Error	$\alpha = 0.9$	$\alpha = 0.8$
−20	−0.219453	−0.219453	3.15588E − 14	−0.219453	−0.219453
−15	−0.219453	−0.219453	6.95128E − 10	−0.219453	−0.219453
−10	−0.219453	−0.219468	1.53112E − 05	−0.219476	−0.219487
−5	−0.219453	−0.554634	3.35181E − 01	−0.718568	−0.936793
5	0.6204960	−0.208263	8.28759E − 01	−0.204179	−0.198958
10	−0.219404	−0.219453	4.86450E − 05	−0.219452	−0.219452
15	−0.219453	−0.219453	2.20851E − 09	−0.219453	−0.219453
20	−0.219453	−0.219453	1.00266E − 13	−0.219453	−0.219453

Fig. 3a showcases the approximate solution for the KK equation. Fig. 3b compares the approximate solution with the exact solution, showing strong agreement for $\alpha = 1$. Lower fractional orders ($\alpha = 0.9, 0.8$) reveal subtle dispersive effects, emphasizing the method's precision in capturing fractional dynamics.

Table 3 evaluates the accuracy of LRPSM across fractional orders and spatial domains. Errors are negligible for $\alpha = 1$ and slightly increase for fractional orders near boundaries. This behavior reflects the nonlocal influences inherent to fractional derivatives.

The findings underscore the efficacy of LRPSM in solving highly nonlinear fractional equations with a high degree of accuracy, even under challenging conditions.



(a) Approximate solution of $u(x, t)$. (b) The approximate solution with the exact solution.

Figure 3: Behaviour of group of 3rd order LRPSM of Eq. (36) at $h = 1$, $k = 1$ where subgraph (a) shows the LRPSM solution of the fractional KK equation and subgraph (b) compares the LRPSM solution with the exact solution for $\alpha = 1$

Table 3: Compare analytical solutions $u(x, t)$ of Eq. (36) obtained by LRPSM with exact for Example 3 at h = 0.3, k = 1

x	Exact	u_{LRPS}	Error	$\alpha = 0.9$	$\alpha = 0.8$
−20	−0.0874061	−0.0874061	5.22540E − 11	−0.0874087	−0.0874109
−15	−0.0785674	−0.0785674	1.71276E − 10	−0.0785788	−0.0785885
−10	−0.0423770	−0.0423770	1.09705E − 10	−0.0424215	−0.042459
−5	0.0683498	0.0683498	2.75872E − 09	0.0682424	0.0681513
0	0.1799520	0.1799520	5.74254E − 09	0.1799420	0.1799330
5	0.0738187	0.0738187	2.75734E − 09	0.0739283	0.0740220
10	−0.0400160	−0.0400160	1.19507E − 10	−0.0399666	−0.0399239
15	−0.0779537	−0.0779537	1.72296E − 10	−0.0779408	−0.0779295

LRPSM consistently produces results with high accuracy, as evidenced by minimal errors across all examples. The method efficiently captures the effects of fractional orders, providing reliable solutions for complex nonlinear systems. The analysis highlights the influence of fractional derivatives in altering wave behavior. Lower fractional orders introduce additional memory effects, resulting in smoother or attenuated waveforms. In fluid mechanics, the influence of α on wave attenuation and dispersion can be applied to model shallow water waves in heterogeneous media, particularly in environments with significant memory effects, such as viscoelastic fluids or stratified flows. This observation aligns with the physical interpretation of fractional dynamics, where historical states significantly affect system evolution. The method demonstrates computational efficiency by converging rapidly to accurate solutions. Compared to traditional numerical approaches, LRPSM provides a more practical and precise alternative for solving fractional differential equations. The soliton solutions derived for each example have significant implications in various fields, including fluid mechanics, plasma physics, and nonlinear optics. By accurately modeling these phenomena, LRPSM facilitates deeper understanding and practical advancements in these domains.

6 Conclusion

The findings of this study underscore the effectiveness of the LRPSM in providing accurate and reliable solutions to fractional fifth-order KdV equations. By successfully addressing the fractional SK, CDG, and KK equations, the method proves its robustness and applicability in modeling complex wave dynamics. The solutions derived using LRPSM have significant physical applications. For instance, they can be used to model soliton propagation in optical fibers, study shallow water waves in fluid mechanics, and analyze ion-acoustic waves in plasma physics. The fractional nature of the equations also makes this method relevant for systems exhibiting memory effects, such as viscoelastic materials and anomalous diffusion in heterogeneous media. Furthermore, the results contribute to advancing wave theory, providing tools for exploring nonlinear optics and quantum mechanics phenomena. This work highlights LRPSM as a versatile and efficient tool for tackling nonlinear fractional differential equations. It paves the way for future research into broader classes of fractional systems and their applications in physics, engineering, and interdisciplinary fields where accurate modeling of wave dynamics is essential.

Acknowledgement: Not applicable.

Funding Statement: This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University, through the Research Funding Program, Grant No. (FRP-1444-12).

Author Contributions: Zuhur Alqahtani and Areej Almuneef conceived of the presented idea. Ahmed Hagag developed the theory, performed the analytic calculations, and performed the numerical simulations. Areej Almuneef verified the analytical methods. Ahmed Hagag drafted manuscript preparation, and supervised the findings of this work. All authors reviewed the results and approved the final version of the manuscript.

Availability of Data and Materials: Not applicable.

Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest to report regarding the present study.

References

1. Elwy O, Abdelaty A, Said L, Radwan A. Fractional calculus definitions, approximations, and engineering applications. J Eng Appl Sci. 2020;67(1):1–30.
2. Jacob JS, Priya JH, Karthika A. Applications of fractional calculus in science and engineering. J Criti Rev. 2020;7(13):4385–94.
3. El-Sayed AM, Arafa A, Hagag A. Mathematical model for the novel coronavirus (2019-nCOV) with clinical data using fractional operator. Numer Methods Partial Differ Equ. 2023;39(2):1008–29. doi:10.1002/num.22915.
4. Stinga PR. Fractional derivatives: fourier, elephants, memory effects, viscoelastic materials, and anomalous diffusions. Notices Am Math Soc. 2023;70(4):576–87. doi:10.1090/noti2663.
5. D'Elia M, Gulian M, Olson H, Karniadakis GE. Towards a unified theory of fractional and nonlocal vector calculus. Fractional Calculus Appl Anal. 2021;24(5):1301–55. doi:10.1515/fca-2021-0057.
6. Sandev T, Tomovski Z, Dubbeldam JL, Chechkin A. Generalized diffusion-wave equation with memory kernel. J Physic A: Mathem Theor. 2018;52(1):015201. doi:10.1088/1751-8121/aaefa3.

7. Metzler R, Klafter J. The random walk's guide to anomalous diffusion: a fractional dynamics approach. Phys Rep. 2000;339(1):1–77. doi:10.1016/S0370-1573(00)00070-3.
8. Dinh TN, Kamal S, Pandey RK. Fractional-order system: control theory and applications. Fract Fractio. 2022;7(1):1–204. doi:10.3390/fractalfract7010048.
9. Alqahtani Z, Hagag AE. A fractional numerical study on a plant disease model with replanting and preventive treatment. Métodos Numéricos Para Cálculo Y Diseño En Ingeniería: Revista Internacional. 2023;39(3):1–21. doi:10.23967/j.rimni.2023.07.001.
10. Arafa AA, Hagag AMS. A different approach for study some fractional evolution equations. Anal Math Phys. 2021;11(4):162. doi:10.1007/s13324-021-00592-3.
11. Barbero G, Evangelista LR, Zola RS, Lenzi EK, Scarfone AM. A brief review of fractional calculus as a Tool for Applications in Physics: adsorption phenomena and electrical impedance in complex fluids. Fract Fractio. 2024;8(7):369. doi:10.3390/fractalfract8070369.
12. Alqahtani Z, Hagag AE. A new semi-analytical solution of compound KdV-Burgers equation of fractional order. Métodos numéricos para cálculo y diseño en ingeniería. Revista Internacional. 2023;39(4):1–16.
13. Khan K, Ali A, Khan ZA, Irfan M. Solitary wave solutions in time-fractional Korteweg-de Vries equations with power law kernel. AIMS Math. 2023;8(1):792–814. doi:10.3934/math.2023039.
14. Yousif MA, Hamasalh FK, Zeeshan A, Abdelwahed M. Efficient simulation of Time-Fractional Korteweg-de Vries equation via conformable-Caputo non-Polynomial spline method. PLoS One. 2024;19(6):e0303760. doi:10.1371/journal.pone.0303760.
15. Akram G, Sadaf M, Sarfraz M, Anum N. Dynamics investigation of (1+1)-dimensional time-fractional potential Korteweg-de Vries equation. Alex Eng J. 2022;61(1):501–9. doi:10.1016/j.aej.2021.06.023.
16. Sawada K, Kotera T. A method for finding N-soliton solutions of the KdV equation and KdV-like equation. Prog Theor Phys. 1974;51(5):1355–67. doi:10.1143/PTP.51.1355.
17. Geyikli T. Collocation method for solving the generalized KdV equation. J Applied Mathematic Physics. 2020;8(6):1123. doi:10.4236/jamp.2020.86085.
18. Miah MM, Meia MAA, Sarker MMR, Hasan A. Adequate closed form wave solutions to the generalized KdV equation in mathematical physics. J Applied Mathematic Physics. 2024;12(6):2069–82. doi:10.4236/jamp.2024.126126.
19. Adivi Sri Venkata RK, Kirubanandam A, Kondooru R. Numerical solutions of time fractional Sawada Kotera Ito equation via natural transform decomposition method with singular and nonsingular kernel derivatives. Math Methods Appl Sci. 2021;44(18):14025–40. doi:10.1002/mma.7672.
20. Yasmin H, Almuqrin AH. Efficient solutions for time fractional Sawada-Kotera, Ito, and Kaup-Kupershmidt equations using an analytical technique. AIMS Math. 2024;9(8):20441–66. doi:10.3934/math.2024994.
21. Fathima D, Alahmadi RA, Khan A, Akhter A, Ganie AH. An efficient analytical approach to investigate fractional Caudrey–Dodd–Gibbon equations with non-singular kernel derivatives. Symmetry. 2023;15(4):850. doi:10.3390/sym15040850.
22. Ganie AH, Mallik S, AlBaidani MM, Khan A, Shah MA. Novel analysis of nonlinear seventh-order fractional Kaup-Kupershmidt equation via the Caputo operator. Boundary Value Probl. 2024;2024(1):87. doi:10.1186/s13661-024-01895-7.
23. Iqbal N, Yasmin H, Rezaiguia A, Kafle J, Almatroud AO, Hassan TS. Analysis of the fractional-order kaup-kupershmidt equation via novel transforms. J Math. 2021;2021(1):2567927. doi:10.1155/2021/2567927.
24. Aycock LM, Hurst HM, Efimkin DK, Genkina D, Lu HI, Galitski VM, et al. Brownian motion of solitons in a Bose-Einstein condensate. Proc Nat Acad Sci. 2017;114(10):2503–8. doi:10.1073/pnas.1615004114.
25. Sakkaravarthi K, Singh S, Karjanto N. Exploring the dynamics of nonlocal nonlinear waves: analytical insights into the extended Kadomtsev-Petviashvili model. Front Phys. 2023;11:1168830. doi:10.3389/fphy.2023.1168830.

26. Almusawa MY, Almusawa H. Numerical analysis of the fractional nonlinear waves of fifth-order KdV and Kawahara equations under Caputo operator. AIMS Math. 2024;9(11):31898–925. doi:10.3934/math.20241533.
27. Hadhoud AR, Rageh AA, Radwan T. Employing the Laplace residual power series method to solve (1+ 1)- and (2+ 1)-dimensional time-fractional nonlinear differential equations. Fract Fractio. 2024;8(7):401. doi:10.3390/fractalfract8070401.
28. Abdelhafeez SA, Arafa AA, Zahran YH, Osman IS, Ramadan M. Adapting Laplace residual power series approach to the Caudrey Dodd Gibbon equation. Sci Rep. 2024;14(1):9772. doi:10.1038/s41598-024-57780-x.
29. Qazza A, Saadeh R. On the analytical solution of fractional SIR epidemic model. Appl Comput Intell Soft Comput. 2023;2023(1):6973734. doi:10.1155/2023/6973734.
30. Ali KK, Abd Elbary FE, Abdel-Wahed MS, Elsisy MA, Semary MS. Solving nonlinear fractional PDEs with applications to physics and engineering using the laplace residual power series method. Int J Differ Equ. 2023;2023(1):1240970. doi:10.1155/2023/1240970.
31. Wang Z, Sun L, Hua R, Su L, Zhang L. Time-fractional generalized fifth-order KdV equation: lie symmetry analysis and conservation laws. Front Phys. 2023;11:1133754. doi:10.3389/fphy.2023.1133754.
32. Ullah I, Ullah A, Ahmad S, Ahmad H, Nofal TA. A survey of KdV-CDG equations via nonsingular fractional operators. AIMS Math. 2023;8(8):18964–81. doi:10.3934/math.2023966.
33. Reddy K, Kumar K, Satish J, Vaithyasubramanian S. A review on applications of Laplace transformations in various fields. J Adv Res Dyn Control Syst. 2017;9(14):14–24.
34. Mainardi F. Fractional calculus: theory and applications. Mathematics. 2018;6(9):145. doi:10.3390/math6090145.
35. Park C, Nuruddeen RI, Ali KK, Muhammad L, Osman MS, Baleanu D. Novel hyperbolic and exponential ansatz methods to the fractional fifth-order Korteweg-de Vries equations. Adv Differ Equ. 2020;2020(1): 1–12. doi:10.1186/s13662-020-03087-w.
36. Karakoc SBG, Saha A, Sucu DY. A collocation algorithm based on septic B-splines and bifurcation of traveling waves for Sawada-Kotera equation. Math Comput Simul. 2023;203(2):12–27. doi:10.1016/j.matcom.2022.06.020.