The 8th European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS Congress 2022
5 – 9 June 2022, Oslo, Norway

MODELING IMPERFECT INTERFACES IN LAYERED BEAMS THROUGH MULTI- AND SINGLE-VARIABLE ZIGZAG KINEMATICS

ROBERTA MASSABÒ, ILARIA MONETTO

* Department of Civil, Chemical and Environmental Engineering
University of Genova
Via Montallegro , 161145 Genova,
e-mail: roberta.bassabo@unige.it, ilaria.monetto@unige.it

Key words: Shear deformable layered beams, homogenization, zigzag kinematics

Abstract. Multiscale structural models based on the coupling of a zigzag kinematics and a cohesive crack approach have been recently formulated to analyze the response of shear deformable layered structures with imperfect interfaces and describe progressive delamination fracture (Massabò, in *Handbook of Damage Mechanics*, Springer, 2022, pp.665-698). The zigzag kinematics accounts for zigzag effects associated to the elastic mismatch of the layers and displacement jumps due to interfacial imperfections, using a limited number of variables, which is independent of the number of layers. The effects of imperfect interfaces on the response of structures subjected to thermo-mechanical loading and on wave propagation and dispersion have been studied analytically and the advantages of this approach over discrete layer models and layerwise theories have been highlighted and discussed. In the presentation we review and discuss these models and present preliminary results on novel single-variable formulations, which have been inspired by a technique developed for homogeneous Timoshenko beams in (Kiendl et al., *Comput Methods Appl Mech Eng*, 284, (2015), 988-1004). An isogeometric collocation scheme is developed for the numerical solution of the problem. The formulation is locking free and satisfies high continuity requirements for the approximating functions.

1 INTRODUCTION

In the last decades composite structures made of layers of the same or different materials, laminated, glued or connected by more or less flexible mechanical devices, have found application in classical and emerging fields, from civil, naval and aeronautical engineering to electronics. In these systems the elastic/thermal mismatch between the layers induces complex distributions of stresses and displacements in the thickness direction, which are also strongly affected by the flexibility of the bonds, their degradation or the presence of delaminations.

The classical equivalent single layer theories developed for layered structures, such as first and higher order shear deformation theories, are inadequate to analyze these systems, due to the assumption of linear, or nonlinear but continuous and smooth, through-thickness variation of the in-plane displacements. The assumption implies that stress continuity is not satisfied at the interfaces between layers made of different materials. The layer-wise theories assume independent displacement fields for each layer, and stress continuity at the interfaces can then be imposed, but requires a high number of degrees of freedom, which is proportional to the number of layers.

The zig-zag theories overcome the drawbacks mentioned above on the basis of a multi-scale approach. The global displacement field of an equivalent single-layer theory is enriched by through-the-thickness zigzag functions. This allows to take into account the effects of the local material inhomogeneities and impose stress continuity at the interfaces; in addition, the homogenization of the displacement field maintains the same degrees of freedom of the base single-layer theory. Recent reviews on the structural theories for layered structures are in [1,2].

Some of the classical zigzag theories are not ideal for finite element implementation, since different continuity requirements are demanded for translational and rotational degrees of freedom; examples are the theories developed in [3] for perfectly bonded layers and in [4] for plates with imperfect interfaces. This, in addition to some inconsistencies and limitations of the zig-zag theories originally formulated for fully bonded systems and subsequently extended to structures with imperfect interfaces and delaminations, has led to the development of refined zig-zag theories [5,6], which however also have some important drawbacks [7,8].

Only recently, the efficacy of the zig-zag approaches was demonstrated also for systems with partially or fully debonded layers. The energetically consistent multiscale model in [9] is based on the original zig-zag theory developed by Di Sciuva [3,10,11] and was formulated for the solution of general plate problems, also with imperfect interfaces, in the presence of static and dynamic mechanical loadings. The model was particularized in [4] to plane-strain problems, extended in [12] to the analysis of plates subjected to thermal loadings, and applied in [13] to study the propagation of plane-strain harmonic waves. Recently, in [14-16], the model was also applied to analyze brittle delamination fracture under mode II dominant conditions in layered beams and wide plates, within the framework of fracture mechanics.

In the presentation we first review the theories based on a zigzag kinematic approximation and formulated for the analysis of structures with imperfect interfaces. Then we explore the possibility of formulating these theories through a novel, single-variable approach, by reference to the model in [12]. The technique, which has been recently developed for homogeneous Timoshenko beams in [17,18], is extended to laminates, where longitudinal and transverse behavior are coupled and can be decoupled only for special stacking sequences and/or selection of the reference surfaces. The reduction of the unknown variables and equations governing the problem facilitates the derivation of analytical solutions, the development of weak forms of the problem, and the implementation of numerical solution procedures, such as those based on the collocation or finite element methods, and can be an efficient alternative approach to the refined zigzag theories. Some promising preliminary results are shown and discussed in the framework of classical first-order shear deformation theory for laminates.

2 STRUCTURAL MULTISCALE MODEL BASED ON ZIGZAG KINEMATIC APPROXIMATION

In this section the formulation of the zig-zag model in [12] is briefly recalled. The multilayered plate of thickness h and length L, in the direction x_2, is assumed to be under plane strain conditions parallel to the plane x_2-x_3 and shown in Fig. 1. In the system of Cartesian coordinates x_1, x_2, x_3, whose origin is arbitrarily placed, the plane x_3=0 defines the reference surface of the plate, which is arbitrarily chosen. The plate consists of n layers joined by n−1 interfaces. Each layer is linearly elastic and orthotropic with principal material axes parallel to the geometrical axes. The interfaces between the layers are zero-thickness mathematical surfaces describing thin elastic interlayers, imperfect bondings and delaminations. The model assumes that only relative sliding displacements can occur at the interfaces whereas relative transverse displacements are not allowed. The interface *k* (with *k*=1,..,*n*−1 numbered from bottom to top) has coordinate x_3^k and defines the upper interface of the layer *k* (with *k*=1,..,*n* numbered from bottom to top) having thickness $^{(k)}h$. The interface is *perfect* when the adjacent layers are fully bonded and no sliding relative displacement can occur; the interface is *imperfect* when relative sliding displacements of the adjacent layers are allowed and related to the interfacial shear tractions through linear interfacial traction laws. Both layers and interfaces can have different mechanical properties.

The plate is subjected to mechanical loads acting on the upper, lower and lateral bounding surfaces, and applied so as to satisfy the plane strain conditions. Transverse normal stresses are set equal to zero ($\sigma_{33} = 0$), since they are negligibly small compared to the other components.

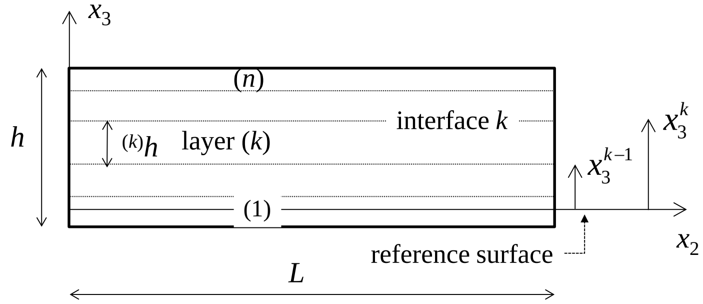

Figure 1: Model geometry

The in plane displacements are assumed as piece-wise linear functions of the through-the-thickness coordinate. The global model is based on classical first-order shear deformation theory and locally enriched in order to reproduce the zig-zag patterns due to the elastic mismatch between the layers and the jumps at the interfaces. The enrichment functions are derived on imposing continuity conditions on the transverse shear stresses and the relationship between interfacial tractions and jumps at the layer interfaces. The local displacement field in the generic layer *k* is then written as follows (comma indicates derivation with respect to x_2)

$$
\begin{aligned}
^{(k)}v_2(x_2,x_3) &= v_{02}(x_2) + x_3\varphi_2(x_2) + [\varphi_2(x_2) + w_{0,2}(x_2)]R^k_{S22}(x_3) \\
^{(k)}v_3(x_2,x_3) &= w_0(x_2)
\end{aligned}
\tag{1}
$$

with

$$
\begin{aligned}
&R^k_{S22}(x_3) = \sum_{i=1}^{k-1}[\Lambda_{22}^{(1;i)}(x_3 - x_3^i) + \Psi^i_{22}]; \quad \Lambda_{22}^{(1;i)} = {}^{(1)}C_{44}\left(1\big/{}^{(i+1)}C_{44} - 1\big/{}^{(i)}C_{44}\right) \\
&\Psi^i_{22} = {}^{(i+1)}C_{44}\Big/K_S^i\left(1+\sum_{j=1}^{i}\Lambda_{22}^{(1;i)}\right) = {}^{(i)}C_{44}\Big/K_S^i\left(1+\sum_{j=1}^{i-1}\Lambda_{22}^{(1;i)}\right)
\end{aligned}
\tag{2}
$$

where: ${}^{(k)}v_i$ is the local displacement in the layer k along the direction x_i with i=2,3 (${}^{(k)}v_1 = 0$ for the assumed plane-strain conditions); v_{02}, w_0 and φ_2 denote the global degrees of freedom which are continuous with continuous derivatives in the thickness direction; Eq. (2) defines the coefficients which account for the local enrichments and depend on the elastic constants of the material (${}^{(k)}C_{44}$ is the stiffness coefficient relating local shear stress and strain in the layer k through ${}^{(k)}\sigma_{23} = {}^{(k)}C_{44}\, 2\, {}^{(k)}\varepsilon_{23}$), the layup and geometry of layers, and on the properties of the interfaces (K_S^k is the interfacial tangential stiffness of the interface k, as defined in Eq. (4)).

From Eq. (1), the local stress field in the generic layer k is obtained through 2D compatibility and constitutive equations

$$\begin{aligned} {}^{(k)}\sigma_{22}(x_2,x_3) &= {}^{(k)}\bar{C}_{22}\{v_{02,2}(x_2) + x_3\varphi_{2,2}(x_2) + [\varphi_{2,2}(x_2) + w_{0,22}(x_2)]R^k_{S22}(x_3)\} \\ {}^{(k)}\sigma_{23}(x_2,x_3) &= {}^{(k)}C_{44}[\varphi_2(x_2) + w_{0,2}(x_2)]\left(1 + \sum_{i=1}^{k-1}\Lambda_{22}^{(1;i)}\right) \end{aligned} \tag{3}$$

where ${}^{(k)}\bar{C}_{22} = {}^{(k)}(C_{22} - C_{23}C_{32}/C_{33})$ relates local longitudinal normal stress and strain through ${}^{(k)}\sigma_{22} = {}^{(k)}\bar{C}_{22}\, {}^{(k)}\varepsilon_{22}$ for the assumed plane-strain conditions, being ${}^{(k)}C_{22}$, ${}^{(k)}C_{23}$, ${}^{(k)}C_{32}$ and ${}^{(k)}C_{33}$ the coefficients of the 6x6 stiffness matrix in the layer k. Finally, from Eq.s (1) and (3) the relative sliding displacement and interfacial tractions at the interface k common to the layers k and k+1 are obtained through the interfacial constitutive law

$${}^{(k)}\sigma_{23}(x_2, x_3 = x_3^k) = {}^{(k+1)}\sigma_{23}(x_2, x_3 = x_3^k) = K_S^k[{}^{(k+1)}v_2(x_2, x_3 = x_3^k) - {}^{(k)}v_2(x_2, x_3 = x_3^k)] \ . \tag{4}$$

The homogenized equilibrium equations are derived using the Principle of Virtual Work and rewritten in terms of global displacements substituting Eq.s (1)-(4)

$$\begin{aligned} &A_{22}v_{02,22} + (B_{22} + C_{22}^{0S})\varphi_{2,22} + C_{22}^{0S}w_{0,222} + f_2 = 0 \\ &(B_{22} + C_{22}^{0S})v_{02,22} + (D_{22} + 2C_{22}^{1S})\varphi_{2,22} + C_{22}^{S2}w_{0,222} - A_{44}(\varphi_2 + w_{0,2}) + f_{2m} = 0 \\ &C_{22}^{0S}v_{02,222} + (C_{22}^{1S} + C_{22}^{S2})\varphi_{2,222} + C_{22}^{S2}w_{0,2222} - A_{44}(\varphi_{2,2} + w_{0,22}) - f_3 = 0 \end{aligned} \tag{5}$$

with coefficients

$$\begin{aligned} &(A_{22}, B_{22}, D_{22}) = \sum_{k=1}^{n}\int_{x_3^{k-1}}^{x_3^k} {}^{(k)}\bar{C}_{22}(1, x_3, x_3^2)\mathrm{d}x_3, \quad C_{22}^{S2} = \sum_{k=1}^{n}\int_{x_3^{k-1}}^{x_3^k} {}^{(k)}\bar{C}_{22}(R^k_{S22})^2\mathrm{d}x_3, \\ &(C_{22}^{0S}, C_{22}^{1S}, C_{22}^{2S}) = \sum_{k=1}^{n}\int_{x_3^{k-1}}^{x_3^k} {}^{(k)}\bar{C}_{22}(1, x_3, x_3^2)R^k_{S22}\mathrm{d}x_3, \quad A_{44} = k_{44}C_{44}^P + C^S, \\ &C_{44}^P = \sum_{k=1}^{n}\int_{x_3^{k-1}}^{x_3^k} {}^{(k)}C_{44}(1 + \sum_{i=1}^{k-1}\Lambda_{22}^{(1;i)})^2\mathrm{d}x_3, \quad C^S = \sum_{k=1}^{n-1}K_s^k(\Psi_{22}^k)^2, \end{aligned} \tag{6}$$

and f_2, f_3 denoting distributed tangential and transverse global load (positive if rightward and upward, respectively) and f_{2m} denoting distributed global couples assumed applied to the reference surface (positive if clockwise); whereas a shear correction factor k_{44} is introduced in A_{44} to improve the treatment of shear given the limitations of the assumed first-order shear deformation theory to describe the global kinematics (refer to [2] for explanations on the need for this coefficient) . The coefficients A_{22} and D_{22} are the extensional and bending laminate stiffnesses, B_{22} is the coupling stiffness of the laminate and relates bending strain with normal force and vice-versa. Analogously, the kinematic and/or static energetically consistent quantities to prescribe at the ends x_2=0, L with outward normal $\mathbf{n}=\{0, \mp 1,0\}^{\mathrm{T}}$ depending on the boundary conditions yield

$$\begin{aligned}
&v_{02} \quad \text{or} \quad N_{22}n_2 = [A_{22}v_{02,2} + (B_{22} + C_{22}^{0S})\varphi_{2,2} + C_{22}^{0S}w_{0,22}]n_2 \\
&w_0 \quad \text{or} \quad Q_{22}n_2 = [-C_{22}^{0S}v_{02,22} - (C_{22}^{1S} + C_{22}^{S2})\varphi_{2,22} - C_{22}^{S2}w_{0,222} + A_{44}(\varphi_2 + w_{0,2})]n_2 \\
&\varphi_2 \quad \text{or} \quad M_{22}n_2 = [(B_{22} + C_{22}^{0S})v_{02,2} + (D_{22} + 2C_{22}^{1S})\varphi_{2,2} + (C_{22}^{1S} + C_{22}^{S2})w_{0,22}]n_2 \\
&w_{0,2} \quad \text{or} \quad M_{22}^{ZS}n_2 = [C_{22}^{0S\,(k)}v_{02,2} + (C_{22}^{1S} + C_{22}^{S2})\varphi_{2,2} + C_{22}^{S2}w_{0,22}]n_2
\end{aligned} \quad . \tag{7}$$

Coupled differential Eq.s (5)-(7) in the three unknowns v_{02}, w_0 and φ_2 govern the general equilibrium problem for multi-layered anisotropic beams and wide plates in cylindrical bending, with imperfect interfaces and according to the energetically consistent zig-zag theory formulated in [12]. When the layers are perfectly bonded (namely, when $1/K_S^k \to 0$ and $\Psi_{22}^k \to 0$ for all k=1,..,n–1 so that C^S=0), these equations coincide with those of first order zigzag models formulated for fully bonded plates.

Finally, when the problem is solved, the local response for each layer and interface is obtained through Eq.s (1)-(4), but for the transverse shear stresses, which are calculated a posteriori by satisfying local equilibrium ${}^{(k)}\sigma_{22,2} + {}^{(k)}\sigma_{23,3}^{post} = 0$.

3 SINGLE VARIABLE FORMULATION

In this section an alternative single-variable formulation of the problem under consideration is presented where the system of coupled differential equations is reduced to a single equation in one unknown variable. The aim is to eliminate some problems, such as that of shear locking, which make zig-zag models not well suited for the implementation in finite element codes and, in general, complicate the derivation of analytical and numerical solutions. As discussed in [17,18] different choices are possible for the primal variable. In this paper, the approach proposed in [18] is followed and the global transverse displacement w_0 is splitted into two parts: a bending part, w_{0b}, and a shear part, w_{0s}, which are defined as follows

$$w_0 = w_{0b} + w_{0s}, \quad \varphi_2 = -w_{0b,2}, \quad \varphi_2 + w_{0,2} = w_{0s,2}\,. \tag{8}$$

In the case of homogeneous Timoshenko beams this special choice allows to express all kinematic and static variables in terms of the new displacement primal variable and its derivatives; this leads to a single differential equation of fourth order accompanied by four boundary conditions which do not contain integrals of the unknown variable.

It is straightforward that Eq.s (5)-(7) are more complicated than those governing the homogeneous beam problem. This is due not only to the presence of local zig-zag enrichments or imperfect interfaces, but also to the coupling between membrane and bending behavior. In order to study the feasibility of extending the single-variable approach to laminates in general conditions, this paper presents a preliminary formulation derived in the framework of first-order shear deformable laminate theory. In order to do this, $R^k_{S22}=0$ is assumed for all k=1,..,n so that $C^{0S}_{22}=C^{1S}_{22}=C^{2S}_{22}=C^{S2}_{22}=C^S=0$, $A_{44}=k_{44}C^P_{44}$ and the equilibrium eq.s in (5) reduce to

$$\begin{aligned}&A_{22}v_{02,22}+B_{22}\varphi_{2,22}+f_2=0\\&B_{22}v_{02,22}+D_{22}\varphi_{2,22}-A_{44}(\varphi_2+w_{0,2})+f_{2m}=0\,.\\&-A_{44}(\varphi_{2,2}+w_{0,22})-f_3=0\end{aligned}\tag{9}$$

Substituting Eq. (8) into the first and second of Eq. (9) yields

$$\begin{aligned}&v_{02,22}=(A_{22})^{-1}(B_{22}w_{0b,222}-f_2),\\&w_{0s,2}=-(A_{44}A_{22})^{-1}\left\{\left[A_{22}D_{22}-\left(B_{22}\right)^2\right]w_{0b,222}+B_{22}f_2-A_{22}f_{2m}\right\},\end{aligned}\tag{10}$$

which can be derived once with respect to x_2 and introduced into the third of Eq. (9) to finally obtain a fourth-order differential equation in the one unknown w_{0b}

$$\left[D_{22}-\left(B_{22}\right)^2(A_{22})^{-1}\right]w_{0b,2222}=f_3-B_{22}(A_{22})^{-1}f_{2,2}+f_{2m,2}\,.\tag{11}$$

When w_{0b} is determined, the global rotation φ_2 follows directly from the second of Eq. (8); the global transverse displacement w_0 follows from the first of Eq. (8) and the second of Eq. (10) as

$$w_0=w_{0b}-(A_{44})^{-1}\left\{\left[D_{22}-\left(B_{22}\right)^2(A_{22})^{-1}\right]w_{0b,22}+\int[B_{22}(A_{22})^{-1}f_2-f_{2m}]\mathrm{d}x_2\right\};\tag{12}$$

the global longitudinal displacement v_{02} follows from the first of Eq. (10) and requires the introduction of two additional arbitrary constants, say c_5 and c_6, as follows

$$v_{02}=B_{22}(A_{22})^{-1}w_{0b,2}-\int\int f_2(A_{22})^{-1}\mathrm{d}x_2\mathrm{d}x_2+c_5x_2+c_6\,.\tag{13}$$

The boundary value problem is completed by six boundary conditions imposed at the ends x_2=0, L with outward normal $\mathbf{n}=\{0,\mp 1,0\}^{\mathrm{T}}$, on the global displacements or on the force and moment resultants energetically consistent with them and defined in the first to third of Eq. (7) specialized for $C^{0S}_{22}=C^{1S}_{22}=C^{2S}_{22}=C^{S2}_{22}=C^S=0$ and $A_{44}=k_{44}C^P_{44}$. In terms of w_{0b} they are

$$
\begin{aligned}
&N_{22}n_2 = [-\int f_2 \mathrm{d}x_2 + A_{22}c_5]n_2, \\
&Q_{22}n_2 = \left\{-[D_{22} - (B_{22})^2(A_{22})^{-1}]w_{0b,222} - B_{22}(A_{22})^{-1} f_2 + f_{2m}\right\} n_2, \\
&M_{22}n_2 = \left\{-[D_{22} - (B_{22})^2(A_{22})^{-1}]w_{0b,22} - B_{22}(A_{22})^{-1} \int f_2 \mathrm{d}x_2 + B_{22}c_5\right\} n_2.
\end{aligned}
\tag{14}
$$

Eq.s (11)-(14) represent a single-variable formulation for laminates under the kinematic constraint of first order shear deformation theory. All kinematic and static variables are expressed in terms of only one unknown and such expressions do not contain integrals of the unknown funtion. This facilitates its employment in discrete numerical solution schemes and excludes a priori locking problems. Furthermore, as for the Bernoulli–Euler beam theory, this formulation is rotation-free, but shear deformability is accounted for. For symmetric laminates, when the reference surface x_3=0 is chosen such that it coincides with the plane of symmetry, B_{22}=0 (x_3=0 is then the neutral plane of the laminate) and Eq.s (11)-(14) reduce to those of the single-variable formulation derived by Kiendl et al. [18] for homogeneous Timoshenko beams, but with f_2 and $f_{2m} \neq 0$. In this case the membrane and bending equilibrium problems are decoupled and the longitudinal displacement v_{02} is independent of w_{0b}.

4 AN ISOGEOMETRIC COLLOCATION SCHEME FOR NUMERICAL MODEL SOLUTION

In this section the laminate problem, governed by the single-variable formulation derived in Section 3, is solved numerically through an isogeometric collocation scheme [19]. This method was introduced in computational mechanics as an alternative to Galerkin-based isogeometric analysis (IGA) [20]. As a collocation approach, it is well suited for the direct solution of problems in the strong form and with boundary conditions containing integrals of the unknowns, such as in the case of displacement-free formulations where rotations are chosen as primal variables [21]. Furthermore, the use of functions developed for Computer Aided Design (CAD) representations, which possess very useful mathematical properties, facilitates the approximation of the unknown variable and satisfies high continuity requirements.

The unknown primal variable w_{0b} is approximated by a B-spline curve, that is a linear combination of m piecewise polynomial basis functions of degree p, say $N_{i,p}$ with i=1,..,m,

$$
w_{0b} \approx \sum_{i=1}^{m} N_{i,p}(\xi = x_2 / L)\hat{w}_{0b_i} = \underline{N}\underline{\hat{w}}_{0b}, \tag{15}
$$

where $\underline{N}$ is the matrix of B-spline shape functions and $\underline{\hat{w}}_{0b}$ is the vector of control variables, which represent the m degreees of freedom of the discrete model. Basis splines (B-splines) are generated through a recursive formula starting with piecewise constants defined on the basis of a so-called knot vector. From Eq. (15) the k-th derivative of the approximated unknown variable can also be approximated as

$$
\frac{\mathrm{d}^k w_{0b}}{\mathrm{d}x_2^k} \approx \sum_{i=1}^{m} \frac{\mathrm{d}^k N_{i,p}(\xi)}{\mathrm{d}\xi^k} L^{-k}\hat{w}_{0b_i} = L^{-k}\underline{N}^{(k)}\underline{\hat{w}}_{0b}. \tag{16}
$$

A knot vector is a set of m+p+1 nondecreasing real numbers (the knots). In the case of 1D straight geometries, such as those under consideration, these numbers represent dimensionless coordinates in the direction x_2, $\xi=x_2/L$. The interval between the first and last knot (the patch) corresponds to the laminate domain which is divided in m+p subintervals (the knot spans). The generation of the basis functions starts from a set of m+p piecewise constant functions, each corresponding to a single knot span where =1 while =0 otherwise, and employs the recursive formula p times. Open knot vectors, where the first and last knots appear p+1 times, are always considered, since this guarantees that basis functions are interpolating at knots located at the ends of the patch, so that boundary conditions can be easily imposed. For further details about B-splines, and their construction and properties, see, e.g., [20,22].

Eq.s (15) and (16) can be used into the second of Eq. (8) and Eq.s (12)-(14) to write all global kinematic and static variables in a discrete form

$$
\begin{aligned}
& w_0 \approx \left\{ \underline{N} - (A_{44})^{-1}\left[D_{22} - \left(B_{22}\right)^2 (A_{22})^{-1} \right] L^{-2} \underline{N}^{''} \right\} \underline{\hat{w}}_{0b} - (A_{44})^{-1} \int [B_{22}(A_{22})^{-1} f_2 - f_{2m}]\mathrm{d}x_2, \\
& \varphi_2 \approx -L^{-1}\underline{N}^{'}\underline{\hat{w}}_{0b}, \\
& v_{02} \approx B_{22}(A_{22})^{-1}L^{-1}\underline{N}^{'}\underline{\hat{w}}_{0b} - \int\int f_2 (A_{22})^{-1}\mathrm{d}x_2\mathrm{d}x_2 + c_5 x_2 + c_6, \\
& Q_{22}n_2 \approx \left\{ -[D_{22} - (B_{22})^2(A_{22})^{-1}]L^{-3}\underline{N}^{'''}\underline{\hat{w}}_{0b} - B_{22}(A_{22})^{-1} f_2 + f_{2m} \right\} n_2, \\
& M_{22}n_2 \approx \left\{ -[D_{22} - (B_{22})^2(A_{22})^{-1}]L^{-2}\underline{N}^{''}\underline{\hat{w}}_{0b} - B_{22}(A_{22})^{-1}\int f_2 \mathrm{d}x_2 + B_{22}c_5 \right\} n_2,
\end{aligned}
\tag{17}
$$

whereas N_{22}, which is independent of w_{0b}, is always given by the first of Eq. (14). Finally, the strong form in Eq. (11) is discretized and evaluated at a set of suitable collocation points, say $\bar{\xi}_i$,

$$
\left[D_{22} - \left(B_{22}\right)^2 (A_{22})^{-1} \right] L^{-4} \underline{N}^{''''}(\bar{\xi}_i)\underline{\hat{w}}_{0b} = f_3(\bar{\xi}_i) - B_{22}(A_{22})^{-1} f_{2,2}(\bar{\xi}_i) + f_{2m,2}(\bar{\xi}_i), \tag{18}
$$

where the so-called Greville abscissae related to the fourth derivative space, that is the order of the differential equation, are chosen as control points [20]. They are defines as follows

$$
\bar{\xi}_i = \frac{\xi_{i+5} + \xi_{i+6} + .. + \xi_{i+p}}{p-4} \quad \text{for } i = 1,..,m-4 . \tag{19}
$$

Eq. (18) gives m–4 equations in the m unknown control variables. The problem is completed by imposing 6 boundary conditions at the ends of the patch ($\bar{\xi}_i = 0,1$) through Eq. (17), which contains the two additional constants c_5 and c_6.

5 APPLICATION TO PAGANO'S STUDY CASE

The single-variable formulation and related numerical scheme developed in Sections 3 and 4 are employed to solve a classical study case, solved by Pagano in closed form in [23]. The cross-ply laminate in Fig. 2 is in cylindrical bending, simply supported at the edges and subjected to a sinusoidal transverse load $f_3(x_2) = q_0 \sin(\pi x_2 / L)$ (f_2=f_{2m}=0). The laminate consists of n=3 unidirectionally reinforced laminae symmetrically arranged with respect to the mid-thickness plane with stacking sequence (0°/90°/0°) and perfectly bonded. Each lamina has thickness t (h=3t) and is orthotropic with elastic moduli E_L, E_T=E_L/25, G_{LT}=E_L/50, G_{TT}=E_L/125, ν_{LT}= ν_{TT}=0.25. The reference surface is chosen to coincide with the bottom of the laminate. This complicates the solution approach, due to the lack symmetry, but allows to validate our formulation for general cases where longitudinal and transverse behaviors cannot be decoupled.

The governing differential equation is of fourth order, then quintic (p=5) and higher degree (p=6) B-splines have been used to discretize the solution. Two meshes, differing not only in the degree of approximating functions but also in the number of degrees of freedom (m=7 for p=5 and m=10 for p=6) have been considered. As an example, Fig. 3 shows the C^4 B-splines used in the coarsest discretization with m=7 and p=5.

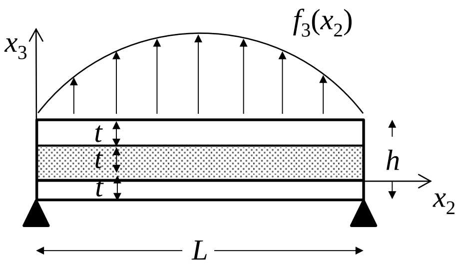

Figure 2: Pagano's study case: geometry and loading condition

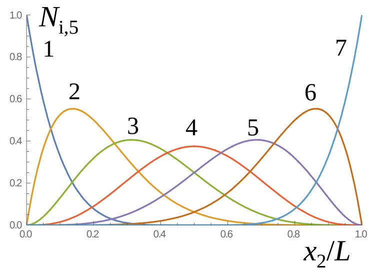

Figure 3: Quintic B-splines generated from the open knot vector $\{0,0,0,0,0,0,0.5,1,1,1,1,1,1\}^T$

Fig.s 4 and 5 show the results obtained for k_{44}=1 and L=4h. In these diagrams the isogeometric collocation based numerical results for discretizations p=5, m=7 (green curves) and p=6, m=10 (blue curves) are shown together with the 1D analytical solution of Eq. (11) (black curves). Fig. 4 shows the dimensionless global transverse displacement and rotation as functions of the dimensionless position along the axis x_2. The analytical and numerical results are in very good agreement, but for the coarsest discretization with p=5 and m= 7. Analogous conclusions can be drawn from Fig. 5, where the dimensionless local longitudinal displacement and in-plane stresses through the thickness are shown and compared with Pagano's 2D exact solution [23] (red curves). The coarsest discretization gives transverse shear stresses which are neither locally in agreement with nor statically equivalent to the 2D exact solution. In general, as expected, laminate theory based solutions do not catch the effective distributions of local kinematic and static variables; a full agreement is expected to be provided by extending the technique to the zigzag model in [12]; this work is in progress.

To conclude, the diagram of through-the-thickness local longitudinal displacements is worth a final comment. The apparent disagreement between 1D structural models and 2D exact solutions follows from imposing the boundary condition $v_{02}=0$ at $\xi=x_2/L=0$. Due to the particular choice made for the reference surface, this boundary condition imposes that the simple support acts at the bottom surface, whereas in Pagano's closed-form solution the simple support is assumed acting at the laminate mid-thickness; in order to compare the two solutions the diagram of ${}^{(k)}v_2$ should be shifted to the right, as shown in Fig. 5 (black dash-dotted curve).

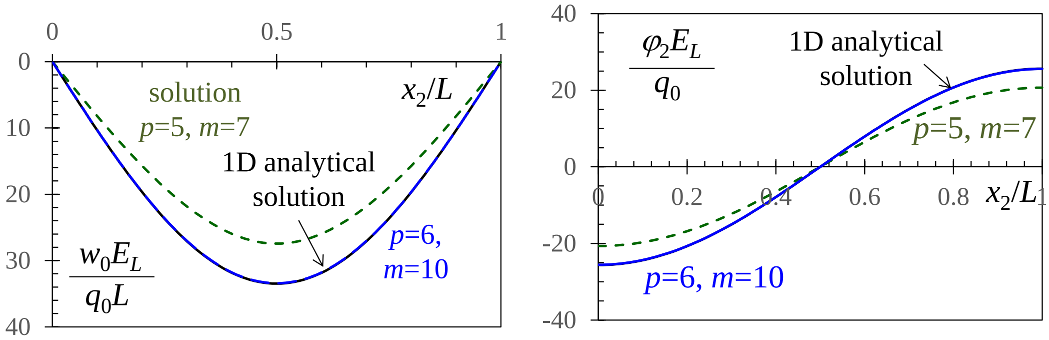

Figure 4: Pagano's study case: global transverse displacement and rotation along the plate length

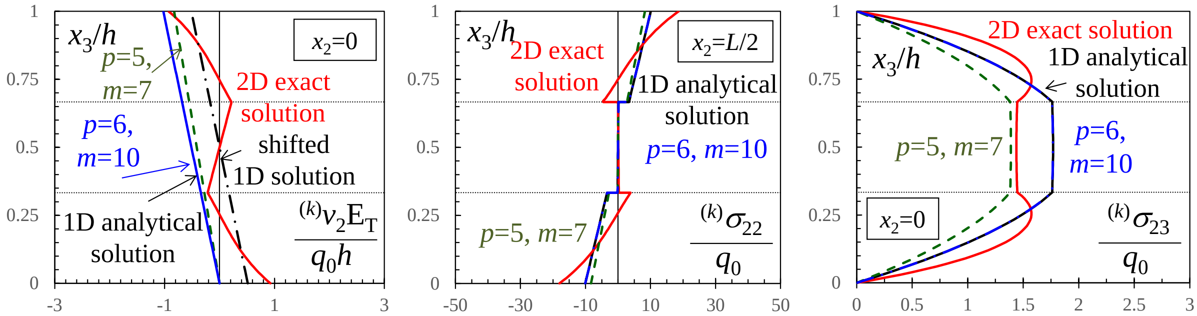

Figure 5: Pagano's study case: local longitudinal displacement and in-plane stresses

6 CONCLUSIONS

- Effective modeling of interlaminar damage in multilayered composite structures using zigzag kinematic approximations and a multiscale approach is reviewed and discussed.
- The multiscale model originally formulated in [4] accurately defines global and local fields in layered beam and plates with interfacial imperfections, subjected to static and dynamic, mechanical and thermal loadings. The model introduces local enrichments (zigzag functions) to the displacement field of a classical first order shear deformation theory and uses a homogenization technique to define the local variables in terms of the global. The problem is then solved using only three (for beams) and five (for plates) global displacement variables, as in classical equivalent single layer theories.
- Preliminary results are presented on a single-variable formulation of the model particularized to beams, which reduces from three to one the displacement unknowns. The feasibility of the approach, which follows and extends that formulated for homogeneous Timoshenko beams in [18], is tested with reference to layered beams

with no interfacial imperfections and neglecting the local enrichments. Under these assumptions the multiscale model coincides with classical first order shear deformation laminate beam theory.

- An isogeometric collocation scheme is developed for the numerical solution of the problem and the technique is applied to a classical case study, Pagano thick three-layered plate subjected to transverse loading.
- The proposed numerical formulation is locking free and satisfies high continuity requirements for the approximating functions.

ACKNOWLEDGEMENTS: support by U.S. ONR and ONR Global, #N62909-21-1-2048 is gratefully acknowledged.

REFERENCES

[1] Abrate, S., Di Sciuva, M.: Multilayer models for composite and sandwich structures. In: Comprehensive Composite Materials II. pp. 399–425. (editors Zweben, C.H., Beaumont, P.W.R.), Elsevier, USA (2018)

[2] R. Massabò, Effective modeling of interlaminar damage in multilayered composite structures using zigzag kinematic approximations, in: G.Z. Voyiadjis (Ed,) Handbook of Damage Mechanics. Springer, New York, 2020, pp. 1-34.

[3] Di Sciuva, M.: Bending, vibration and buckling of simply supported thick multilayered orthotropic plates: An evaluation of a new displacement model. J. Sound Vib. 105(3), 425–442 (1986).

[4] R. Massabò, F. Campi, Assessment and correction of theories for multilayered plates with imperfect interfaces, Meccanica. 50 (2015) 1045-1071, DOI: 10.1007/s11012-014-9994-x.

[5] Tessler, A., M. di S.: Refinement of Timoshenko Beam Theory for Composite and Sandwich Beams Using Zigzag Kinematics. Natl. Aeronaut. Sp. Adm. NASA/TP-2007-215086. (2007)

[6] Tessler, A., Di Sciuva, M., Gherlone, M.: A refined zigzag beam theory for composite and sandwich beams. J. Compos. Mater. 43(9), 1051–81 (2009).

[7] Darban, H., Massabò, R.: A homogenized structural model for shear deformable composites with compliant interlayers. Multiscale Multidiscip. Model. Exp. Des. 1, 269–290 (2018).

[8] Flores, F.G., Oller, S., Nallim, L.G.: On the analysis of non-homogeneous laminates using the refined zigzag theory. Compos. Struct. 204, 791–802 (2018)

[9] R. Massabò, F. Campi, An efficient approach for multilayered beams and wide plates with imperfect interfaces and delaminations, Composite structures. 116 (2014) 311-324

[10] M. Di Sciuva, M., An improved shear-deformation theory for moderately thick multilayered anisotropic shells and plates, J. Appl. Mech. 54 (1987) 589-596.

[11] M. Di Sciuva, Geometrically nonlinear theory of multilayered plates with interlayer slips, AIAA J. 35 (1997) 1753-1759.

[12] M. Pelassa, R. Massabò, Explicit solutions for multi-layered wide plates and beams with perfect and imperfect bonding and delaminations under thermo-mechanical loading, Meccanica. 50 (2015) 2497-2524.

[13] R. Massabò, Propagation of Rayleigh-Lamb waves in multilayered plates through a multiscale structural model, Int. J. Solids and Structures. 124 (2017) 108-124.

[14] Massabò, R., Influence of boundary conditions on the response of multilayered plates with cohesive interfaces and delaminations using a homogenized approach, (2014) Frattura ed Integrita Strutturale, 8 (29), pp. 230-240. DOI: 10.3221/IGF-ESIS.29.20

[15] R. Massabò, H. Darban, Mode II dominant fracture of layered composite beams and wide-plates: a homogenized structural approach, Eng. Fracture Mech. 213 (2019) 280-301.

[16] R. Massabò, I. Monetto, Local zigzag effects and brittle delamination fracture of n-layered beams using a structural theory with three displacement variables, Frattura ed Integrità Strutturale. 51 (2020) 275-287, DOI: 10.3221/IGF-ESIS.51.22.

[17] G. Falsone, D. Settineri, An Euler–Bernoulli-like finite element method for Timoshenko beams, Mechanics Research Communications. 38 (2011) 12-16.

[18] J. Kiendl, F. Auricchio, T.J.R. Hughes, A. Reali, Single-variable formulations and isogeometric discretizations for shear deformable beams, Comput. Methods Appl. Mech. Engrg. 284 (2015) 988-1004.

[19] F. Auricchio F., L. Beirão da Veiga, T. J. R. Hughes, A. Reali, G. Sangalli, Isogeometric collocation methods, Math. Models Methods Appl. Sci. 20 (2010) 2075-2107.

[20] T.J.R. Hughes, J.A. Cottrell, Y. Bazilevs, Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, Comput. Methods Appl. Mech. Engrg. 194 (2005) 4135-4195.

[21] J. Kiendl, F. Auricchio, A. Reali, A displacement-free formulation for the Timoshenko beam problem and a corresponding isogeometric collocation approach, Meccanica. 53 (2018) 1403-1413.

[22] Cottrell, J.A., Hughes, T.J.R. and Bazilevs, Y. Isogeometric analysis: Towards Integration of CAD and FEA. Wiley (2005).

[23] N.J. Pagano, Exact solutions for composite laminates in cylindrical bending, J. Compos. Mater. 3 (1969) 398-411.