

NEW FORMULATION OF METALLIC MATERIALS CONSTITUTIVE MODELS BY AN ENERGY APPROACH

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Abstract. The literature proposes a wide class of constitutive laws modeling the thermomechanical behavior of materials, expressing the stress tensor in function of the instantaneous internal or local variables generally represented by the strain tensor, strain rate one and temperature. From a thermodynamic point of view, the stress undergone by the material is rather related to the dissipated plastic strain energy responsible for any change in the microstructure state of the material. In this work a new formalism for defining hardening or more general constitutive laws as a function of the energy dissipated by the plastic

deformation, called energetic approach, is proposed. This new formalism makes it possible to integrate all the physical mechanisms which govern the materials plasticity and change of the material microstructure. This formulation remains valid for a large field of deformations and it can accurately predict the material plastic flow undergoing all types of loadings: static, transient or dynamic. One of the main objectives of this work is to give a more general mathematical formulation of the constitutive plastic laws concerning metals hardening and softening. The new constitutive model based on proposed energetic approaches will be discussed comparing to a numerical incremental resolution of the Prandtl-Reuss equations.

1 INTRODUCTION

In the mechanics of plastic materials flow and structures strength, various thermomechanical behavior models are proposed by a lot of authors, in order to properly predict the material response during manufacturing and forming process especially when subjected to severe deformations and complex paths. The classical constitutive laws describing the thermomechanical material behavior express, especially in the context of elastoplasticity, the Cauchy stress of a continuous deformable material according to elastic and plastic deformation, strain rate, temperature. Phenomenological laws ([1]-[4]) use the evolution of macroscopic internal variables, indirectly based on physical phenomena but neglecting some evolutions or description of material microscopic changes. In contrast, pure physical-based laws ([5]-[7]) take into account thermodynamics principles and kinetic equations describing the microstructure evolution especially based on the dislocations dynamics. However, this is done by adding more materials parameters to be identified. These classical behaviour formulations, reduce the effect of the real strain and strain rate history. Yet, these laws are specific to each type of material, depending on the scale and the range of considered deformation. Since behaviour equations must be more general and physically based on thermodynamics principles, consequently it is important to construct constitutive models that characterize an already “material identity card” describing rheological flow.

According to Hill [8], during a plastic flow the dissipated specific plastic deformation energy W is an essential internal variable especially during a forming process. This energy can be estimated from a defined equivalent stress $\bar{\sigma}$ and cumulative plastic strain $\bar{\epsilon}_p$ by:

$$W = \int \bar{\sigma} d\bar{\epsilon}_p \quad \text{where} \quad dW = \bar{\sigma} d\bar{\epsilon}_p \quad (1)$$

Hence, inspired by the models of elastic [8] or hyper-elastic behavior ([9]-[11]) formulations based on energy potentials, an energy approach as that proposed by Yoshino and Shirakashi [12] gives, in the framework of a purely plastic behavior, a formulation of the equivalent Cauchy stress as a function of the specific plastic deformation energy $\bar{\sigma}(W)$. In a physical sense the deformation energy is seen as the most representative internal variable characterising the material state evolution state. Previous works relies the energy approach from general physical principles (Thermodynamics Laws, Virtual Work Principle, Plasticity criteria...). In this scientific paper will be presented concisely the proposed general energy approach giving the equivalent stress defined from a plastic criterion (as Von-Mises, Hill or other one) in terms of the specific plastic deformation energy. Thus, it will first be proven for simple rheological laws that the energetic constitutive law formulation is equivalent to these ones. Then, this approach can be applied to classical hardening laws expressing the equivalent

stress function of cumulative plastic deformation $\bar{\sigma}(\bar{\epsilon}_p)$ and especially in the case of more general formulations as those proposed by previous works of Gavrus [6] taking into account strong coupling between dynamic recovery and softening phenomena. Constitutive differential equations models including energy approach will be then developed. The energetic approach will be also developed to try to solve elastoplastic Prandtl-Reuss equations.

2 GENERAL FRAMEWORK

As stated by Yoshino and Shirakashi in [12] an energetic formulation of plastic behavior laws try to express the Cauchy stress tensor $\underline{\underline{\sigma}}$ as a function of the specific plastic energy W . Using this formulation the obtained new constitutive material laws take into account the deformation history characterising the plastic material flow verifying especially the objectivity principle¹. Furthermore the specific plastic deformation energy take into account the material deformation history and includes the influence of physical plastic deformation phenomena on the material microstructure evolution. As it is known, in a classical way the metallic materials hardening behaviour laws are expressed for a specific strain-rate and temperature as follows:

$$\bar{\sigma} = f(\bar{\epsilon}_p) \quad (2)$$

Here $\bar{\sigma}$ is the equivalent stress defined in the case of isotropy by the Von-Mises loci $\sqrt{\frac{3}{2} \underline{\underline{\sigma'}} : \underline{\underline{\sigma'}}$ with $\underline{\underline{\sigma'}} = \underline{\underline{\sigma}} + p$ and $p = -\frac{1}{3} \text{Trace}(\underline{\underline{\sigma}})$ where "·" is the matrix contraction product) and $\bar{\epsilon}_p$ is the cumulative plastic deformation defined by the integral of the generalized plastic strain rate $\dot{\bar{\epsilon}}_p$ defined from the plastic strain rate tensor $\underline{\underline{\dot{\epsilon}}}_p = \underline{\underline{\dot{\epsilon}}} - \underline{\underline{\dot{\epsilon}}}_{el}$ by the formula $\bar{\epsilon}_p = \int_0^t \dot{\bar{\epsilon}}_p = \int_0^t \sqrt{\frac{2}{3} \underline{\underline{\dot{\epsilon}}}_p : \underline{\underline{\dot{\epsilon}}}_p}$ (where $\underline{\underline{\dot{\epsilon}}}$ is the total strain tensor and $\underline{\underline{\dot{\epsilon}}}_{el}$ is the elastic strain tensor part).

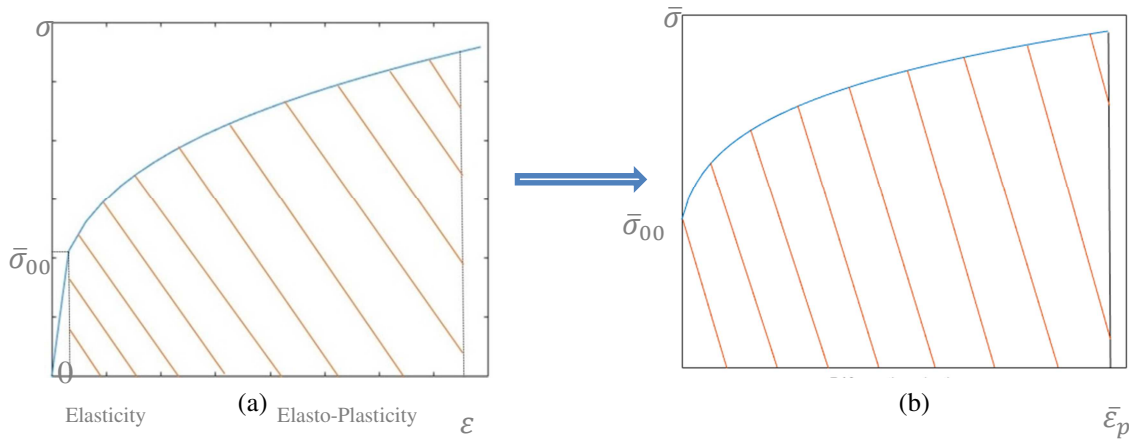


Figure 1: a) The true stress – total true strain flow curve obtained from an uniaxial tensile or pure compression test (where σ is the true uniaxial loading stress term and $\epsilon = \int_0^t \dot{\epsilon}$ is the true uniaxial total strain term expressed in function of the corresponding total strain rate term $\dot{\epsilon}$), b) True stress-plastic strain flow curve ($\bar{\sigma}_{00}$ represents the yield stress value).

Starting from a classical tensile test, the Figure 1 shows the variation of the principal Cauchy tensor composant σ defined along the axial traction direction, named also true stress,

¹ The parameters introduced must be independent of the selected referential

with respect to the corresponding eulerian total deformation ε and its conversion in a true stress-true plastic strain curve described in a classical manner by the constitutive equation (2).

By reversing Eq. (2) and by differentiating with respect to $\bar{\sigma}$, we get:

$$\frac{df^{-1}(\bar{\sigma})}{d\bar{\sigma}} = \frac{d\bar{\varepsilon}_p}{d\bar{\sigma}} \quad (3)$$

Considering that $g(\bar{\sigma}) = \bar{\sigma} \frac{df^{-1}(\bar{\sigma})}{d\bar{\sigma}}$, the latter equation can be written as:

$$g(\bar{\sigma})d\bar{\sigma} = \bar{\sigma}d\bar{\varepsilon}_p \quad (4)$$

Integrating equation (4) and recalling that $W = \int \bar{\sigma}d\bar{\varepsilon}_p$ it is obtained:

$$W = G(\bar{\sigma}) \quad (5)$$

Finally, the energetic law expressing equivalent stress in terms of the specific plastic deformation energy is expressed by:

$$\bar{\sigma} = H(W) \text{ where } H = G^{-1} \quad (6)$$

The Figure 2 illustrates this conversion of the classical true stress - true plastic strain flow curve into an energetic one using the definition of the specific plastic energy (1) represented by the hatched part (Fig. 2a).

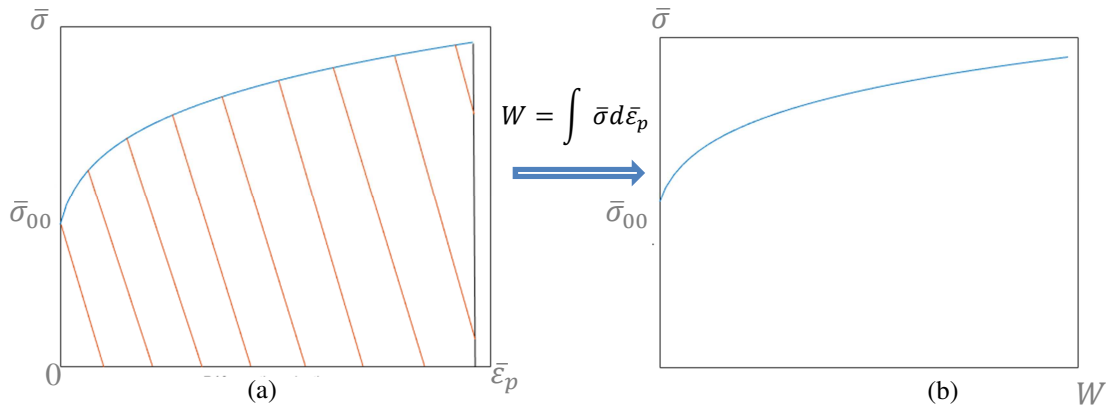


Figure 2: True stress – true plastic strain flow curve (a) converted into the true stress – plastic deformation energy flow curve (b) (energy approach).

The inverse approach, *i.e.* retrieving classical rheological law formulation from the energetic one is aborted in [12]. Thus by reversing Eq.(6) it can be found:

$$H^{-1}(\bar{\sigma}) = \int \bar{\sigma}d\bar{\varepsilon}_p \quad (7)$$

The derivative of eq.(7) with respect to $\bar{\varepsilon}_p$ leads to:

$$\frac{\partial H^{-1}(\bar{\sigma})}{\partial \bar{\sigma}} \frac{\partial \bar{\sigma}}{\partial \bar{\varepsilon}_p} = \bar{\sigma} \quad (8)$$

$$\frac{1}{\bar{\sigma}} \frac{\partial H^{-1}(\bar{\sigma})}{\partial \bar{\sigma}} \partial \bar{\sigma} = d\bar{\varepsilon}_p \quad (9)$$

By integrating the latter equation, it can be written:

$$\int \frac{1}{\bar{\sigma}} \frac{\partial H^{-1}(\bar{\sigma})}{\partial \bar{\sigma}} \partial \bar{\sigma} = \bar{\varepsilon}_p \Leftrightarrow \bar{\varepsilon}_p = F(\bar{\sigma}) \quad (10)$$

So it is obtained the known classic law:

$$\bar{\sigma} = f(\bar{\varepsilon}_p) \text{ where } f = F^{-1} \quad (11)$$

It can be deduced the equivalence between the clasical formulation and the energetic one:

$$\bar{\sigma}(\bar{\varepsilon}_p) \Leftrightarrow \bar{\sigma}(W) \quad (12)$$

This energy approach will be adopted for a lot of classical rheological relationships describing the flow behavior of metallic materials in the following section.

3 CLASSICAL RHEOLOGICAL LAWS AND ENERGETIC APPROACH

The main objectif is to develop an energy approach for a lot of traditional laws modelizing the work hardening plastic flow of metallic materials. Thus, we can find energetic laws, expressing plastic strain as a function of dissipated deformation energy. First, this approach is established for the widely known Hollomon power law [13], Swift or Krupkowski law [14], Ludwik law [14], classical Voce law [15] and generalized Voce law [6].

3.1 Hollomon Power Law

The Hollomon power law [13] describes a simple hardening effect defined by the increase of the equivalent stress with the plastic deformation and is expressed in the following form:

$$\bar{\sigma} = K(\bar{\varepsilon}_p)^n \quad (13)$$

where K and n are the respectively consistency and hardening parameters of the material.

This law gives stress only as a function of plastic deformation. Reversing this law to obtain the expression of cumulative plastic deformation $\bar{\varepsilon}_p$ and differentiating the obtained expression with respect to equivalent stress $\bar{\sigma}$, yields:

$$\frac{1}{n} \left(\frac{\bar{\sigma}}{K} \right)^{1/n} \frac{1}{\bar{\sigma}} d\bar{\sigma} = d\bar{\varepsilon}_p \quad (14)$$

By integrating equation (14), $\bar{\sigma}$ is expressed in terms of the energy W as follows:

$$\int \frac{1}{n} \left(\frac{\bar{\sigma}}{K} \right)^{1/n} d\bar{\sigma} = \int \bar{\sigma} d\bar{\varepsilon}_p \text{ i.e. } \bar{\sigma} = K' W^{n'} + C \quad (15)$$

where K' and n' are given by :

$$K' = K^{\frac{1}{n+1}} (n+1)^{\frac{n}{n+1}} \Leftrightarrow K = K'^{\frac{1}{1-n'}} (1-n')^{\frac{n'}{1-n'}} \text{ and } n' = \frac{n}{n+1} \Leftrightarrow n = \frac{n'}{1-n'} \quad (16)$$

Regarding eq. (13), for $\bar{\varepsilon}_p = 0$ it is obtained $\bar{\sigma} = W = 0$ and consequently the integrating constant $C = 0$. The associated energetic expression of Hollomon power law is written by:

$$\bar{\sigma} = K' W^{n'} \quad (17)$$

3.2 Swift Power Law

In the same way the Swift or Krupkowski power law [14] is expressed as:

$$\bar{\sigma} = K(\bar{\varepsilon}_p + \bar{\varepsilon}_0)^n \quad (18)$$

where $\bar{\varepsilon}_0$ is an initial plastic deformation corresponding to the yield point $\bar{\sigma}_{00} = K\bar{\varepsilon}_0^n$.

Expressing $\bar{\varepsilon}_p$ from eq.(18) in function of $\bar{\sigma}$ and differentiating with respect to $\bar{\sigma}$ it is find :

$$-\frac{1}{n} \left(\frac{\bar{\sigma}}{K} \right)^{1/n} d\bar{\sigma} = \bar{\sigma} d\bar{\varepsilon}_p \quad (19)$$

Now by integrating the latter equation, we obtain the corresponding energetic law as:

$$\bar{\sigma} = K' \left[W + \frac{K}{n+1} (\bar{\varepsilon}_0)^{n+1} \right]^{n'} \quad (20)$$

where K' and n' are defined by the same relationships as in formula (16).

3.3 Ludwik Law

Classical Ludwik law [14] takes into consideration the yield stress $\bar{\sigma}_{00}$ in an additive form and is described as follows:

$$\bar{\sigma} = \bar{\sigma}_{00} + K(\bar{\varepsilon}_p)^n \quad (21)$$

This law models the nonlinear increasing part of the plastic hardening flow curve pictured in Figure 2a. Equation (21) can be equivalently written as:

$$\left(\frac{\bar{\sigma} - \bar{\sigma}_{00}}{K} \right)^{1/n} = \bar{\varepsilon}_p \quad (22)$$

The differentiation of the obtained equation with respect to $\bar{\sigma}$ gives:

$$\frac{1}{nK^{1/n}} (\bar{\sigma} - \bar{\sigma}_{00})^{1/n} d\bar{\sigma} = (\bar{\sigma} - \bar{\sigma}_{00}) d\bar{\varepsilon}_p \quad (23)$$

By integration of previous equation it is obtained:

$$\frac{1}{(n+1)K^{1/n}} (\bar{\sigma} - \bar{\sigma}_{00})^{\frac{n+1}{n}} = W - W_0 + C' \quad (24)$$

Here $W_0 = \int \bar{\sigma}_{00} d\bar{\varepsilon}_p = \bar{\sigma}_{00} \bar{\varepsilon}_p$ represents a reference plastic energy related to yield stress.

Finally, the energetic law may be written in the following form:

$$\bar{\sigma} = \bar{\sigma}_{00} + K'(W - W_0 + C')^{n'} \quad (25)$$

$$n' = \frac{n}{n+1} \leftrightarrow n = \frac{n'}{1-n'} \quad \text{and} \quad K' = K^{\frac{1}{n+1}} (n+1)^{\frac{n}{n+1}} \leftrightarrow K = K'^{\frac{1}{(1-n')}} (1-n')^{\frac{n'}{1-n'}} \quad (26)$$

As can be regarded in the eq. (21), for $\bar{\varepsilon}_p = 0$ it is find $W_0(0) = 0$ and $\bar{\sigma}(0) = \bar{\sigma}_{00}$. Hence the energetic law is established as follows:

$$\bar{\sigma} = \bar{\sigma}_{00} + K'(W)^{n'} \quad (27)$$

3.4 Classical Voce Law

The classical Voce law [15] describing a hardening-recovery plastic flow is defined by:

$$\bar{\sigma} = \bar{\sigma}_{00} + K(1 - e^{-n\bar{\varepsilon}_p}) \quad (28)$$

First, the plastic strain from eq. (28) is expressed as follows:

$$-\frac{1}{n} \ln(1 - \frac{\bar{\sigma} - \bar{\sigma}_{00}}{K}) = \bar{\varepsilon}_p \quad (29)$$

The derivative of the last equation with respect to $\bar{\sigma}$ gives:

$$\frac{1}{n(K + \bar{\sigma}_{00} - \bar{\sigma})} d\bar{\sigma} = \bar{\sigma} d\bar{\varepsilon}_p \quad (30)$$

It can be easily shown that integrating equation (30) gives:

$$A - \bar{\sigma} - A \ln(A - \bar{\sigma}) = nW + K - A \ln(K) \text{ with } A = K + \bar{\sigma}_{00} \quad (31)$$

Using equation (31) and variable change $u = \frac{\bar{\sigma} - A}{A}$ (with $-1 \leq u \leq 0$) it can be written:

$$u \exp(u) = -\frac{K}{A} \exp\left(-\frac{nW+K}{A}\right) \quad (32)$$

The analytical solution of the previous equation is given by:

$$u = Z_0 \left[-\frac{K}{A} \exp\left(-\frac{nW+K}{A}\right) \right] \quad (33)$$

Here $Z_0(\cdot)$ is the Lambert function² chosen because $u = Z_k(\cdot) \geq -1$. This function does not have any explicit form but is recognised using Matlab along the graph pictured in Figure 3. Hence, it can find the energetic Voce law form:

$$\bar{\sigma} = A \left\{ 1 + Z_0 \left[-\frac{K}{A} \exp\left(-\frac{nW+K}{A}\right) \right] \right\} \quad (34)$$

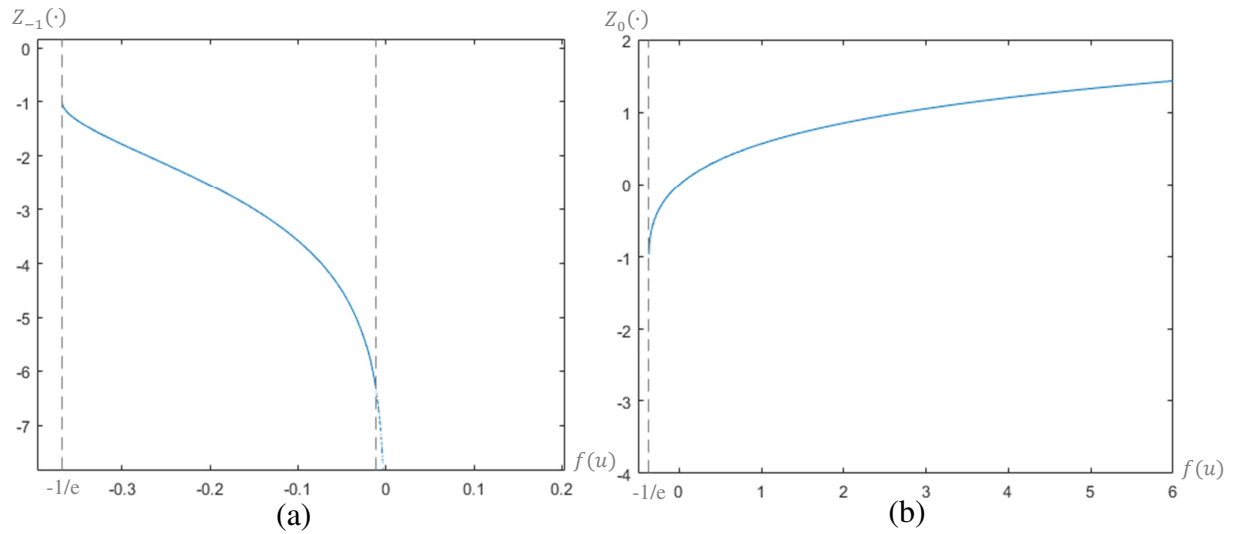


Figure 3: Graphic representation of the Lambert function: a) noted Z_{-1} for $-1/e \leq f(u) < 0$ with $Z_k(\cdot) \leq -1$.
b) noted Z_0 for $-1/e \leq f(u) < +\infty$ with $Z_k(\cdot) \geq -1$.

From fig. 3 it is possible to observe that $-\infty < Z_k(\cdot) < 2$ for real entries. A lot of particular values of Lambert function is given in Table 1.

Table 1: Particular values of the Lambert function Z_0 and Z_{-1} .

$f(u)$	$-1/e$	-0.35	-0.3	0.25	-0.2	-0.15	-0.1	-0.05	0
$u=Z_0(\cdot)$	-1	-0.7166	-0.4894	-0.3574	-0.2592	-0.1795	-0.1118	-0.0527	0
$f(u)$	0.5	1	1.5	2	2.5	3	3.5	4	4.5
$u=Z_0(\cdot)$	0.352	0.5671	0.7259	0.8526	0.9586	1.0499	1.1303	1.2022	1.2672
$f(u)$	$-1/e$	-0.35	-0.3	0.25	-0.2	-0.15	-0.1	-0.05	0
$u=Z_{-1}(\cdot)$	-1	-1.3497	-1.7813	-2.1533	-2.5426	-2.9936	-3.5772	-4.4998	$-\infty$

² Lambert function $Z_k(\cdot)$ [16] is the inverse form of $f(u) = u \exp(u)$ to determine $u = Z_k(\cdot) = f^{-1}(u)$ and is defined on $[-1/e; +\infty]$.

3.5 Generalised Voce Law

The Generalized Voce Law poposed by Gavras ([1], [6]) is expressed by:

$$\bar{\sigma} = \bar{\sigma}_{00} + K(1 - e^{-n\bar{\varepsilon}_p})^{n_a} \quad (35)$$

where here n_a is a hardening parameter.

This law is not limited only to relatively small plastic deformation ($< 0.3-0.5$) like the classical one and can be used for a various class of metallic materials undergoing dynamic recovery with values of n_a function of metals cristalographic system [6] . In order to obtain the corresponding energetic formulation first is find the the plastic strain $\bar{\varepsilon}_p$:

$$-\frac{1}{n} \ln(1 - (\frac{\bar{\sigma} - \bar{\sigma}_{00}}{K})^{\frac{1}{n_a}}) = \bar{\varepsilon}_p \quad (36)$$

Using $N = nn_a$, $A = (K + \bar{\sigma}_{00})$ and differentiating this expression with respect to $\bar{\sigma}$ is given:

$$\frac{1}{N} \frac{\bar{\sigma}(\bar{\sigma} - \bar{\sigma}_{00})^{\frac{1-n_a}{n_a}}}{1 - (\frac{\bar{\sigma} - \bar{\sigma}_{00}}{K})^{\frac{1}{n_a}}} d\bar{\sigma} = \bar{\sigma} d\bar{\varepsilon}_p \quad (37)$$

Using the variable change $u = \bar{\sigma} - \bar{\sigma}_{00}$ the integration of eq.(37) gives:

$$-\bar{\sigma}_{00} \ln \left[K^{\frac{1}{n_a}} - (\bar{\sigma} - \bar{\sigma}_{00})^{\frac{1}{n_a}} \right] + \frac{K^{-\frac{1}{n_a}}(\bar{\sigma} - \bar{\sigma}_{00})^{\frac{n_a+1}{n_a}}}{n_a+1} {}_2F_1 \left(1, 1+n_a, 2+n_a, K^{-\frac{1}{n_a}}(\bar{\sigma} - \bar{\sigma}_{00})^{\frac{1}{n_a}} \right) = nW \quad (38)$$

where $F_1(a, b, c, x)$ is the Hypergeometric function³ defined as follows:

$$F_1 \left(1, 1+n_a, 2+n_a, K^{-\frac{1}{n_a}}(\bar{\sigma} - \bar{\sigma}_{00})^{\frac{1}{n_a}} \right) = \sum_{n'=0}^{\infty} \frac{(1)_{n'}(1+n_a)_{n'}}{(2+n_a)_{n'}} \frac{[K^{-\frac{1}{n_a}}(\bar{\sigma} - \bar{\sigma}_{00})^{\frac{1}{n_a}}]^{n'}}{n'!} \quad (39)$$

Concerning this law, the energetic formulation does not have any analytical expression. Numerical solution can be finding using for example an Euler schema. An energetic formulation based on differentials equations models will be proposed in following section 4.5.

3.6 General Constitutive Law Formulation

Starting form the general constitutive model formulation proposed by Gavras ([1],[6]) coupling hardening, dynamic recovery and softening toghether with behaviour description at various ranges of plastic strain, strain-rate and temperatures, it can be written for a reference plastic flow curve (in standard conditions of strain-rate and at ambient temperature) by:

$$\bar{\sigma} = \bar{\sigma}_{00} + \bar{\sigma}_h(\bar{\varepsilon}_p)[1 - \psi(\bar{\varepsilon}_p)] + \psi(\bar{\varepsilon}_p)\bar{\sigma}_s \quad (40)$$

Here $\bar{\sigma}_h(\bar{\varepsilon}_p)$ is the hardening function which can be chosen from classical laws synthetised in Table 2, $\psi = 1 - e^{-r\bar{\varepsilon}_p^s}$ is the generalized Avrami function proposed in [6] to describe material fraction udergoing softening phenomena (where r et s are the corresponding softening parameters) and $\bar{\sigma}_s$ is the saturation stress. The derivative of the Avrami fraction ψ with respects to $\bar{\varepsilon}_p$ leads to:

$$\frac{d\psi}{d\bar{\varepsilon}_p} = -rs\bar{\varepsilon}_p^{s-1}(1 - \psi) = -rs \left[\ln(1 - \psi)^{-\frac{1}{r}} \right]^{\frac{s-1}{s}} (1 - \psi) \quad (41)$$

³ Hypergeometric function [17] is defined for $|x| < 1$ as: $F_1(a, b, c, x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!}$. Where the symbol $(a)_{n=0} = 1$ and $(a)_{n>0} = a(a+1) \dots (a+n-1)$.

Table 2: Hardening function term

Law	$\bar{\sigma}_h(\bar{\varepsilon}_p)$
Hollomon	$K\bar{\varepsilon}_p^n$
Swift or Krupkowski	$K(\bar{\varepsilon}_p + \bar{\varepsilon}_0)^n$
Ludwik	$\bar{\sigma}_{00} + K(\bar{\varepsilon}_p)^n$
Classical Voce	$\bar{\sigma}_{00} + K(1 - e^{-n\bar{\varepsilon}_p})$
Generalized Voce	$\bar{\sigma}_{00} + K(1 - e^{-n\bar{\varepsilon}_p})^{n_a}$

Starting from the equivalence proven in previous paper section illustrated by Eq. (12), the energetic formulation of this law can be given by:

$$\bar{\sigma} = \bar{\sigma}_{00} + \bar{\sigma}_h(W)(1 - \psi(W)) + \psi(W)\bar{\sigma}_s \quad (42)$$

where $\bar{\sigma}_h(W)$ is the hardening stress expressed in terms of energy W (based as the previous developments concerning classical hardening laws) and $\psi(W)$ is the energetic form of Avrami fraction. More details will be presented in below paper section 4.6.

4 GENERAL ENERGETIC DIFFERENTIAL CONSTITUTIVE MODEL

This part search to develop energetic differential constitutive equations for each particular previous law in order to try to find a general energetic model.

4.1 Hollomon power law

In order to obtain the associated differential energetic model for the power hardening law this one can be written in the form:

$$\bar{\sigma} = K\varphi \quad \text{with} \quad \varphi = (\bar{\varepsilon}_p)^n \quad (43)$$

Firstly, by differentiating this law with respect to W , it is found:

$$\frac{d\bar{\sigma}}{dW} = K \frac{d(\varphi)}{dW} \quad (44)$$

On the other hand, the derivative of φ with respect to W , gives the second differential equation of the model:

$$\frac{d\varphi}{dW} = \frac{d\varphi}{d\bar{\varepsilon}_p} \frac{d\bar{\varepsilon}_p}{dW} = \frac{n}{K\varphi} \varphi^{\frac{n-1}{n}} = \frac{n}{K} \varphi^{\frac{-1}{n}} \quad (45)$$

Hence, by adding initial conditions, the following differential equations system is deduced:

$$\begin{cases} \frac{d\bar{\sigma}}{dW} = K \frac{d(\varphi)}{dW} \\ \varphi^{\frac{1}{n}} \frac{d\varphi}{dW} = \frac{n}{K} \\ \bar{\sigma}(0) = \bar{\sigma}_{00} = 0 \\ \varphi(0) = 0 \end{cases} \quad (46)$$

The analytical solutions of this model are given by:

$$\varphi(W) = \left(\frac{W(n+1)}{K} \right)^{\frac{n}{n+1}} \quad (47)$$

$$\bar{\sigma}(W) = K \left(\frac{W(1+n)}{K} \right)^{\frac{n}{n+1}} = K'W^{n'} \quad (48)$$

The same analytical solution as the energetic law (17) is obtained.

4.2 Swift Power Law

The constitutive equations for the Swift or Krupkowski law can be written as:

$$\bar{\sigma} = K\varphi \text{ with } \varphi = (\bar{\varepsilon}_p + \bar{\varepsilon}_0)^n \quad (49)$$

By differentiating $\bar{\sigma}$ with respect to W once get:

$$\frac{d\bar{\sigma}}{dW} = K \frac{d(\varphi)}{dW} \quad (50)$$

On the other side, the derivative of φ with respect to energy W gives:

$$\frac{d\varphi}{dW} = \frac{d\varphi}{d\bar{\varepsilon}_p} \frac{d\bar{\varepsilon}_p}{dW} = \frac{n}{K} \varphi^{\frac{-1}{n}} \quad (51)$$

By adding initial conditions, the energetic differential equations of this law are well established through the differential system:

$$\begin{cases} \frac{d\bar{\sigma}}{dW} = K \frac{d(\varphi)}{dW} \\ \varphi^{\frac{1}{n}} \frac{d\varphi}{dW} = \frac{n}{K} \\ \bar{\sigma}(0) = \bar{\sigma}_{00} = K(\bar{\varepsilon}_0)^n \\ \varphi(0) = (\bar{\varepsilon}_0)^n \end{cases} \quad (52)$$

Analytical solutions of this differential system are obtained as:

$$\varphi(W) = \left(\frac{W(n+1)}{K} + \bar{\varepsilon}_0^{n+1} \right)^{\frac{n}{n+1}} \quad (53)$$

$$\bar{\sigma}(W) = K^{\frac{1}{n+1}} (W(1+n) + K\bar{\varepsilon}_0^{n+1})^{\frac{n}{n+1}} \quad (54)$$

4.3 Ludwik Law

The Ludwik law can be expressed by:

$$\bar{\sigma} = \bar{\sigma}_{00} + K\varphi \text{ with } \varphi = (\bar{\varepsilon}_p)^n \quad (55)$$

Differentiating this law with respect to W yields:

$$\frac{d\bar{\sigma}}{dW} = K \frac{d(\varphi)}{dW} \quad (56)$$

Furthermore, differentiating φ with respect to the energy W gives:

$$\frac{d\varphi}{dW} = \frac{d\varphi}{d\bar{\varepsilon}_p} \frac{d\bar{\varepsilon}_p}{dW} = \frac{n}{K} \varphi^{\frac{-1}{n}} \quad (57)$$

Therefore the differential system of this model is expressed as:

$$\begin{cases} \frac{d\bar{\sigma}}{dW} = K \frac{d(\varphi)}{dW} \\ \varphi^{\frac{1}{n}} \frac{d\varphi}{dW} = \frac{n}{K} \\ \bar{\sigma}(0) = \bar{\sigma}_{00} \\ \varphi(0) = 0 \end{cases} \quad (58)$$

The solutions of this system are given by:

$$\varphi(W) = \left(\frac{W(n+1)}{K} \right)^{\frac{n}{n+1}} \quad (59)$$

$$\bar{\sigma}(W) = \bar{\sigma}_{00} + K'W^{n'} = \bar{\sigma}_{00} + K^{\frac{1}{n+1}}(n+1)^{\frac{n}{n+1}}W^{\frac{n}{n+1}} \quad (60)$$

4.4 Classical Voce Law

The classical Voce law can be written in the form:

$$\bar{\sigma} = \bar{\sigma}_{00} + K\varphi \quad \text{whith} \quad \varphi = 1 - \exp(-n\bar{\varepsilon}_p) \quad (61)$$

By differentiating this law with respect to W the first and second equations become:

$$\frac{d\bar{\sigma}}{dW} = K \frac{d(\varphi)}{dW} \quad (62)$$

$$\frac{d\varphi}{dW} = \frac{d\varphi}{d\bar{\varepsilon}_p} \frac{d\bar{\varepsilon}_p}{dW} = \frac{n(1-\varphi)}{\bar{\sigma}} = \frac{n(1-\varphi)}{\bar{\sigma}_{00} + K\varphi} \quad (63)$$

Hence, the differential form of energetic constitutive model is given by:

$$\begin{cases} \frac{d\bar{\sigma}}{dW} = K \frac{d(\varphi)}{dW} \\ \frac{(\bar{\sigma}_{00} + K\varphi)}{(1-\varphi)} \frac{d\varphi}{dW} = n \\ \bar{\sigma}(0) = \bar{\sigma}_{00} \\ \varphi(0) = 0 \end{cases} \quad (64)$$

The solutions are obtained in a similar form than eq. (34) as:

$$\bar{\sigma} = \bar{\sigma}_{00} + K + (K + \bar{\sigma}_{00}) \mathbf{Z}_0 \left[-\frac{\frac{-(K+nW)}{K + \bar{\sigma}_{00}}}{K + \bar{\sigma}_{00}} \right] \quad (65)$$

4.5 Generalized Voce Law

In a similar way can be established the associated energetic model corresponding to the Generalized Voce Law:

$$\bar{\sigma} = \bar{\sigma}_{00} + K\varphi \quad \text{with} \quad \varphi = (1 - e^{-n\bar{\varepsilon}_p})^{n_a} \quad (66)$$

Differentiating this law with respect to W yields:

$$\frac{d\bar{\sigma}}{dW} = K \frac{d(\varphi)}{dW} \quad (67)$$

Now, by differentiating φ with respect to W , it is given:

$$\frac{d\varphi}{dW} = \frac{d\varphi}{d\bar{\varepsilon}_p} \frac{d\bar{\varepsilon}_p}{dW} \quad (68)$$

The second equation of this model is developed as following:

$$(\bar{\sigma}_{00} + K\varphi) \frac{\varphi^{1-n_a}}{(1-\varphi^{1/n_a})} \frac{d\varphi}{dW} = N \quad (69)$$

Finally, the energetic system is given by:

$$\begin{cases} \frac{d\bar{\sigma}}{dW} = K \frac{d(\varphi)}{dW} \\ (\bar{\sigma}_{00} + K\varphi) \frac{\varphi^{1-n_a}}{(1-\varphi^{1/n_a})} \frac{d\varphi}{dW} = N \\ \bar{\sigma}(0) = \bar{\sigma}_{00} \\ \varphi(0) = 0 \end{cases} \quad (70)$$

The solution of this problem is given by numerical methods. The second differential equation of the system (74) can be written as:

$$\frac{d\varphi}{dW} = \frac{N}{(\bar{\sigma}_{00} + K\varphi)} \frac{1-\varphi^{1/n_a}}{\varphi^{1-n_a}} = f(W, \varphi(W)) \quad (71)$$

where $f(W, \varphi(W))$ is a continuous function from $\mathbb{R}^+ \times U$ in $\mathbb{R}^d$.

Now, by using an explicit Euler method with an incremental formulation using a constant energetic step defined by $\Delta W = W^{k+1} - W^k$ and $W^k = W^0 + k\Delta W$ (where k is an entire number defining the increment), the solution can be approached as follows:

$$\varphi(W^{k+1}) = \varphi(W^k) + \Delta W f(W^k, \varphi(W^k)) \quad (72)$$

Here $f(W^k, \varphi(W^k))$ is the function value at previous increment “ k ”. Once known the value of φ it is possible to define the solution $\bar{\sigma}$ from the first differential equation in (70) integrated analytically and then find the numerical solution of this differential system.

4.6 Differential Equations for General Constitutive Law Formulation

To obtain an energetic constitutive model of eq. (40) it is derivate with respect to W :

$$\frac{d\bar{\sigma}}{dW} = (1 - \psi) \frac{d\sigma_h(W)}{dW} - \sigma_h(W) \frac{d\psi}{dW} + \sigma_s \frac{d\psi}{dW} \quad (73)$$

On the other hand, using eq. (41) the derivative of Avrami fraction ψ with respect to the specific plastic energy W gives:

$$\frac{d\psi}{dW} = \frac{d\psi}{d\bar{\epsilon}_p} \frac{d\bar{\epsilon}_p}{dW} \quad (74)$$

$$\bar{\sigma} \frac{d\psi}{dW} = -rs \left[\ln(1 - \psi)^{-\frac{1}{r}} \right]^{\frac{s-1}{s}} (1 - \psi) \quad (75)$$

The following differential equation's constitutive model is finding:

$$\begin{cases} \frac{d\bar{\sigma}}{dW} = (1 - \psi) \frac{d\sigma_h(W)}{dW} - \sigma_h(W) \frac{d\psi}{dW} + \sigma_s \frac{d\psi}{dW} \\ \bar{\sigma} \frac{d\psi}{dW} = -rs \left[\ln(1 - \psi)^{-\frac{1}{r}} \right]^{\frac{s-1}{s}} (1 - \psi) \\ \bar{\sigma}(0) = \bar{\sigma}_{00} + \bar{\sigma}_{h0} \\ \psi(0) = 0 \end{cases} \quad (76)$$

Here $\sigma_h(W)$ can be derived from one of the previous energetic differentials equations characterizing hardening laws. The differential system can be solved using an Euler schema.

5 INCREMENTAL ENERGETIC PRANDTL-REUSS MODEL RESOLUTION

This section will be dedicated for the energetic approach of the Prandtl-Reuss elasto-plastic model. Starting form the expression of the total strain-rate tensor $\underline{\underline{\dot{\epsilon}}}$ as the sum of the elastic $\underline{\underline{\dot{\epsilon}}}_{el}$ and plastic stain-rate $\underline{\underline{\dot{\epsilon}}}_p$ [18] it can be written:

$$\underline{\underline{\dot{\epsilon}}} = \underline{\underline{\dot{\epsilon}}}_{el} + \underline{\underline{\dot{\epsilon}}}_p \quad (77)$$

Using the double contraction product between the Cauchy stress $\underline{\underline{\sigma}}$ and the strain rate $\underline{\underline{\dot{\epsilon}}}$ defined by the latter equation, it can be find the total specific deformation power $\dot{W}$ as the sum of the specific elastic energy power $\dot{W}_{el}$ and the specific plastic energy power as follows:

$$\dot{W} = \dot{W}_{el} + \dot{W}_p \quad (78)$$

The elastic part of the total power $\dot{W}$ is given by :

$$\dot{W}_{el} = \underline{\underline{\sigma}} : \underline{\underline{\dot{\sigma}}} \quad (79)$$

were $\underline{\underline{D}}$ is the elastic stiffness fourth-order tensor.

Moreover, to define the plastic part of the energy power, the normal rule law of the plastic strain rate for a defined plasticity criteria $\bar{\sigma} = F(\underline{\underline{\sigma}})$, using a scalar plastic multiplier λ_p , given:

$$\underline{\underline{\dot{\epsilon}}}_p = \lambda_p \frac{\partial F(\underline{\underline{\sigma}})}{\partial \underline{\underline{\sigma}}} \quad (80)$$

The plastic part of the total power $\dot{W}_p = \underline{\underline{\sigma}} : \underline{\underline{\dot{\epsilon}}}_p$ can be expressed then by:

$$\dot{W}_p = \lambda_p \underline{\underline{\sigma}} : \frac{\partial F(\underline{\underline{\sigma}})}{\partial \underline{\underline{\sigma}}} \quad (81)$$

Thus the total power $\dot{W}$ is computed as follows :

$$\dot{W} = \underline{\underline{\sigma}} : \underline{\underline{\dot{\epsilon}}} + \lambda_p \underline{\underline{\sigma}} : \frac{\partial F(\underline{\underline{\sigma}})}{\partial \underline{\underline{\sigma}}} \quad (82)$$

Particulary, for a Von-Mises isotropic plastic criterion, the latter expression becomes:

$$\dot{W} = \underline{\underline{\sigma}} : \underline{\underline{\dot{\epsilon}}} + \frac{3\lambda_p}{2\bar{\sigma}} \underline{\underline{\sigma}} : \underline{\underline{\dot{\sigma}}} \quad (83)$$

The elastic deformation power is composed from deviatoric and hydrostatic parts i.e.:

$$\dot{W}_{el} = \underline{\underline{\sigma}}' : \underline{\underline{\dot{\sigma}}}' + \underline{\underline{\sigma}}_s : \underline{\underline{\dot{\sigma}}}_s \quad (84)$$

Here $\underline{\underline{\sigma}}'$ is the deviatoric stress and $\underline{\underline{\sigma}}_s$ is the diagonal hydrostatic stress.

In order to find the expression of λ_p , the total power $\dot{W}$ is expressed in terms of deviatoric stress $\underline{\underline{\sigma}}'$ only due to its relevance in plasticity phase and can be espressed as:

$$\dot{W} = \dot{W}_{el} + \dot{W}_p = \underline{\underline{\sigma}}' : \underline{\underline{\dot{\sigma}}}' + \lambda_p \underline{\underline{\sigma}}' : \frac{\partial F'(\underline{\underline{\sigma}}')}{\partial \underline{\underline{\sigma}}'} \quad (85)$$

where for a Von-Mises yield criteria $F'(\underline{\underline{\sigma}}') = \underline{\underline{\sigma}}' : \underline{\underline{\sigma}}' - \frac{2}{3} \bar{\sigma}^2(\bar{\epsilon}_p) = 0$ (for simplicity reasons).

By substituting $\bar{\sigma}(\bar{\epsilon}_p)$ by $\bar{\sigma}(W_p)$ using energetic approach, the Eq. (85) can be rewritten using isotropic elasticity as:

$$\dot{W} = \frac{1}{2\mu} \underline{\underline{\sigma}}' : \underline{\underline{\dot{\sigma}}}' + 2\lambda_p \underline{\underline{\sigma}}' : \underline{\underline{\dot{\sigma}}}' \quad (86)$$

The derivative of $\underline{\underline{\sigma}}' : \underline{\underline{\dot{\sigma}}}'$ with respect to time t give:

$$\frac{d}{dt} (\underline{\underline{\sigma}}' : \underline{\underline{\dot{\sigma}}}') = 2 \underline{\underline{\sigma}}' : \underline{\underline{\dot{\sigma}}}' = \frac{4}{3} \bar{\sigma}(W_p) \frac{d\bar{\sigma}}{dW_p} \frac{dW_p}{dt} \quad (87)$$

In the other hand, using the Equation (86) it is obtained:

$$\underline{\underline{\sigma}}' : \underline{\underline{\dot{\sigma}}}' = 2\mu(\dot{W} - 2\lambda_p \underline{\underline{\sigma}}' : \underline{\underline{\dot{\sigma}}}') \quad (88)$$

So, using the equations (87) and (88) with $\frac{dW_p}{dt} = \frac{2}{3} \lambda_p \bar{\sigma}^2$ and $\underline{\underline{\sigma}}' : \underline{\underline{\dot{\sigma}}}' = \frac{2}{3} \bar{\sigma}^2(W_p)$ is find the following expression:

$$2\mu (\dot{W} - 2\lambda_p \underline{\underline{\sigma}}' : \underline{\underline{\dot{\sigma}}}') = \frac{4}{9} \lambda_p \bar{\sigma}(W_p) \frac{d\bar{\sigma}}{dW_p} \bar{\sigma}^2 \quad (89)$$

Hence, the plastic multiplier λ_p is obtained as follows:

$$\lambda_p = \frac{\dot{W}}{\frac{2}{3} \bar{\sigma}^2 (\frac{1}{3\mu} \frac{d\bar{\sigma}}{dW_p} + 2)} \quad (90)$$

6 CONCLUSIONS

The main aim of this work was to develop an energetic approach for a new formulation of metals constitutive models. First have been established energetic laws corresponding to classical plastic flow relationships. For these laws the energetic differential equations models give exactly same energetic form as the analytical one. This allowed validation of proposed differential equations constitutive models to be used in a more general way. A specific energetic approach has been proposed to solve Prandtl-Reuss elasto-plastic equations.

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