Fundamental Theorems and Wave Dynamics in Rotating Thermoelastic Diffusion with Multi-Phase Delays and Temperature-Dependent Properties

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ABSTRACT

The current study uses a novel mathematical formulation of temperature-dependent thermoelastic diffusion with multi-phase delays in a rotating frame of reference (TDMR). Higher order time derivatives are included for the diffusing mass flux, the gradient of chemical potential, the gradient of temperature, and the TDMR model of heat flow vector in Fourier's and Fick's laws. The fundamental theorems (energy, uniqueness, reciprocity, and variational criterion) are examined using the preliminary equations for the simulated model TDMR. Reciprocity theorem applications are taken for particular scenarios including body forces, heat sources, and chemical potential sources. It has been noted that the variational criteria and these theorems are mostly influenced by the field variables' susceptibility and the variations in the parameters of higher-order temporal derivatives. The two-dimensional example of the assumed model's planar wave propagation is also provided. Four connected longitudinal waves have been identified: the primary (P) wave, secondary (S) wave, thermal (T) wave, and chemical potential (CP) wave. Wave properties such as phase velocity and attenuation coefficient are estimated numerically and displayed visually. A few special examples are also investigated and connected with the established outcomes. This work provides a foundation for further research into basic issues in thermoelastic continua under various physical field factors. Numerous applications in material science, geomechanics, soil dynamics, and the electronic sector can be made of the current findings.

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1 Introduction

The fundamental principles of heat conduction and diffusion are based on the works of Fourier [1] and Fick [2]. To resolve the conflict between linked and uncoupled theories, generalized theories of thermoelasticity have been developed over the past four decades. A generalized thermoelastic model with one relaxation time was proposed by Lord and Shulman [3] (LS), whereas Green and Lindsay [4] (GL) corrected the LS model by adding another relaxation time to the corresponding equations and created a new theory of thermoelasticity. With this change, the GL model may now rely only on the strain-acceleration relationship. Thermal displacement gradient was introduced to the humanistic Fourier law by Green and Naghdi [5] to evaluate the movement of heat without energy dissipation. In the work of Hetnarski and Ignaczak [6], numerous studies grounded in generalized theories of thermoelasticity have been taken into consideration.

Tzou [7] presented phase lag with heat flow initially, and he generalized it in the Cattaneo-Vernotte [8,9] (CV) formulation. The most common phrase for this concept is the single-phase-lag heat transfer hypothesis. Tzou [10,11] developed the dual-phase-lag (DPL) heat transfer theory by collecting the temperature gradient and heat flux using two-phase-lag time constants. The three-phase-lag thermoelastic model, which Roychoudhari [12] introduced, is an extension of the Dual-phase-lag (DPL) theory. It replaces Fourier's law of heat conduction with an approximation of an alteration to Fourier's law by taking into account three distinct phase lags for the heat flux vector, the temperature gradient, and the thermal displacement gradient Tzou [13] provided extensive work on phase delays and thermal conductivity in his book.

Based on coupled thermoelastic theory, Nowacki [14–17] developed the humanistic thermoelastic diffusion model. Sherief et al. [18] and Kumar and Kansal [19] have suggested the thermoelastic diffusion theory with one and two relaxation durations, respectively. Aouadi studied thermoelastic diffusion materials with vacancies [20]. The majority of issues in extended thermoelasticity theory were accepted with the assumption that material properties remain constant with temperature, which restricts the application of the obtained solutions to specific temperature ranges. But it will soon come for systems supporting situations with variable temperatures, where certain material characteristics—like elastic modulus, thermal conductivity, Poisson's ratio, thermal expansion coefficient, thermal diffusivity, and chemical potential, among others—become temperature-dependent rather than constant [21]. Therefore, considering temperature-dependent features makes more sense when examining the generalized thermoelastic issues' dynamic antiphon. Ezzat et al. [22] looked into a problem where the temperature-dependent modulus of elasticity was examined to examine how temperature-dependent characteristics affect the behaviors of problems in thermoelastic theories. Under the notion of fractional order, Abouelregal [23] presented a boundary value problem in a semi-infinite piezoelectric material with temperature-dependent features. A fractional-order generalized thermoelasticity problem for a half-space with variable thermal conductivity was studied by Youssef and Ibrahim [24]. Under the thermoelasticity theory, Adel et al. [25] examined a photo-thermo-piezoelectric issue for orthotropic rotational semiconductors with moisture diffusion and laser excitation.

Reciprocity and uniqueness in anisotropic thermoelastic diffusion materials were given by Aouadi [26]. Kuang [27] developed the variational criterion and created the governing equations for generalized thermodiffusion theory with pyroelectricity. Based on the isotropic thermoelastic diffusion model, Li et al. [28] established the thermoelastic diffusion of porous elastic materials with some application in the ultrashort pulse laser heating. The uniqueness reciprocity theorem and wave phenomena in thermoelastic diffusion with fractional derivative, as well as in several theories of thermoelasticity, were provided by Kumar and Gupta [29,30]. Energy and uniqueness theorems were introduced by Sarkar and Mukhopadhyay [31] in conjunction with the variational criterion in a thermoelastic model featuring memory-dependent derivatives (MDD) and three-phase delays.

Basic theorems (uniqueness, continuous dependence, and reciprocity) in relaxed meromorphic thermoelastic material were investigated by Bala [32]. The reciprocity theorem and variational criterion were investigated by Shivay and Mukhopadhyay [33] in poro-thermoelastic materials with temperature-dependent characteristics. Al-Lehaibi [34] investigated thermoelastic materials at two temperatures using the variational criterion without energy dissipation. Uniqueness, reciprocity, and variational criteria in thermoelastic diffusion models were introduced by Paul and Mukhopadhyay [35] along with a novel mathematical model of generalized thermoelastic diffusion theory.

Abouelregal [36–38] has made several recent attempts to alter the humanistic Fourier law of heat conduction by considering time derivatives of higher order. The improved multi-phase-lag (MPL) model discovers numerous benefits in a range of issues. A different version of the RPL model was presented by Zenkour [39–41] to address the thermoelastic responses of numerous different composite materials. Thermoelastic diffusion with fractional order derivatives was constructed by Ezzat and Fayik [42], who also demonstrated variational criteria, uniqueness, and reciprocity theorems. A few theorems in piezo-thermoelastic with dual-phase-lag and piezo-thermoelastic with the fractional order derivatives with two temperatures were analyzed [43–45]. Semiconducting excitation processes with thermal relaxation times were presented according to the photothermal transport process [46,47].

Different authors developed different thermoelastic models to investigate distinct wave analysis difficulties. Plane waves and the fundamental solution in the expanded theories of thermoelastic diffusion were derived [48]. The propagation of plane waves and fundamental solution in homogeneous, isotropic electro-microstretch elastic solids and electro-microstretch viscoelastic solids were studied by Sharma et al. [49]. The plane waves and fundamental solution in a modified pair stress generalized thermoelastic with mass diffusion were studied by Kumar et al. [50]. Sharma and Khator [51,52] looked into a few issues related to producing electricity from renewable sources.

The diffusion, rotation, and lagging behavior of thermoelastic micropolar media with voids and a temperature gradient under mechanic pressure were established by Hilal [53]. A thermoelastic model with higher order time derivatives for a crack in a rotating material was studied by Panja et al. in [54]. The propagation and reflection of thermoelastic waves under Hall current influence in a rotating nonlocal fractional order porous medium were established by Bibi et al. [55]. Rotation and viscosity's effects on generalized conformable fractional micropolar thermoelasticity at two temperatures were studied by El-Sapa et al. [56]. The study of thermoelasticity has been significantly advanced through various mathematical approaches, including energy methods and variational principles. Marin et al. [57] investigated the asymptotic equipartition of energies in initially stressed bodies, providing important insights into energy distribution in thermoelastic systems. Additionally, the Lagrange identity method has been effectively applied to microstretch thermoelastic materials to establish fundamental identities and reciprocity relations [58]. Further advancements in the field include the application of the Relaxed Saint-Venant principle to thermoelastic micropolar diffusion, which refines our understanding of stress distribution in complex materials [59]. These studies provide a strong theoretical foundation for the present work, which extends the analysis to multi-phase-lag thermoelastic diffusion with rotation and temperature-dependent properties. Recent advances in thermoelastic diffusion have increasingly focused on integrating more comprehensive theoretical frameworks to address the limitations of classical models. For instance, Abouelregal and Sedighi [60] offer new insights into the interaction of thermoelasticity with mass diffusion in a half-space by employing the Moore–Gibson–Thompson thermodiffusion theory, which provides a refined description of delay effects in heat and mass transport processes. Their work not only broadens the understanding of non-Fourier heat conduction and diffusion phenomena but also serves as a compelling complement to traditional approaches. In our study, we build on these ideas by further incorporating rotational dynamics and temperature-dependent properties into a multi-phase-lag thermoelastic framework, thereby enriching the modeling capabilities and expanding the scope of applications in complex materials and geophysical systems.

While numerous studies have explored thermoelastic diffusion, phase-lag effects, and temperature-dependent properties, most of these works assume a stationary frame of reference, thereby overlooking the unique influences of rotation. In our study, we develop a comprehensive model that integrates multi-phase-lag thermoelastic diffusion with temperature-dependent properties within a rotating coordinate system. This novel approach enables us to capture inertial effects, such as Coriolis and centripetal accelerations, which significantly alter wave propagation characteristics, energy distribution, and attenuation behavior phenomena that previous models could not adequately explain. By addressing these aspects, our work fills a critical gap in the literature, offering new insights particularly relevant for applications in rotating machinery, geomechanics, and advanced material systems. This study presents a novel theoretical formulation of thermoelastic diffusion with multi-phase lags and temperature-dependent properties in a rotating reference frame. Unlike previous studies that consider constant material properties or simpler phase-lag models, our approach incorporates:

1. A refined multi-phase-lag model that extends traditional single- and dual-phase-lag theories by considering additional delay effects in heat conduction and diffusion.
2. The influence of rotation and temperature-dependent material properties on wave propagation has not been extensively analyzed in previous works.
3. A rigorous formulation of fundamental theorems (uniqueness, energy, reciprocity, and variational criteria) that ensures mathematical consistency and provides a foundation for future studies.
4. A comprehensive numerical analysis of phase velocities and attenuation coefficients for multiple wave modes (P-wave, S-wave, thermal wave, and chemical potential wave), offering new insights into wave behavior in thermoelastic diffusion media.

The fundamental theorems of energy, uniqueness, reciprocity, and variational principle have all been established using this model. Additionally, plane propagation in the two-dimensional case of the assumed temperature-dependent thermoelastic diffusion with multi-phase delays in a rotating frame of reference (TDMR) model is provided. The attenuation coefficient and phase velocity of waves are estimated numerically and displayed visually. A few special situations are inferred and contrasted with the established findings.

2 Governing Equations

In thermoelastic diffusion under the multi-phase-lag model, consider the domain V of three-dimensional space ($\mathbb{R}^3$), which is limited by the piecewise smooth surface A. Let t $\in \overline{V} \times [0, \infty)$ and $\mathbf{x} = (x_1, x_2, x_3)$ be the positions of the state variables, where $\overline{V} = V \cup A$. The governing equations for stress-strain relations and motion are derived from the generalized theory of thermoelastic diffusion. The stress-strain relations, equation of motion, and energy balance equations are then modified accordingly to include rotational effects and temperature-dependent properties. The final governing equations are presented as follows:

Relation of stress, strain, temperature, and chemical potential [3,4,20]:

$$t_{ij} = 2\mu e_{ij} + \delta_{ij}\left(\lambda_0 e_{kk} - \gamma_1 T - \gamma_2 P\right) \tag{1}$$

Equation of motion [3–6,18]:

$$t_{ij,j} + \rho F_i = \rho \ddot{u}_i + \rho(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i + \rho(2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})_i \tag{2}$$

Eq. (1) represents the constitutive relation based on Hooke's law modified for thermoelastic diffusion, while Eq. (2) is the equation of motion incorporating rotational effects.

Displacement-strain relation:

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

The governing equations for thermoelastic diffusion with multi-phase lags in a rotating medium are developed by extending the classical thermoelasticity framework. In the conventional model, heat conduction follows Fourier's law, and mass diffusion is described by Fick's law. However, these models assume instantaneous propagation of thermal and diffusion effects, which contradicts experimental observations.

To overcome this limitation, we incorporate the multi-phase-lag (MPL) model, which introduces higher-order time derivatives into the heat flux and mass diffusion equations. The MPL model accounts for delayed responses in heat and mass transfer by incorporating three distinct phase lags:

1. Phase lag of heat flux (τ_q) represents the delay in heat propagation.

2. Phase lag of temperature gradient (τ_T) that accounts for non-Fourier effects in heat conduction.

The generalized heat conduction equation incorporating MPL effects is given by

Energy equation [20]:

$$-\,q_{i,i} + Q = \rho T_0 \dot{S} \tag{4}$$

Modified Fourier law [7,8]:

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!}\frac{\partial^r}{\partial t^r}\right) q_i = -K\left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!}\frac{\partial^r}{\partial t^r}\right) T_{,i} \tag{5}$$

The classical Fourier law assumes an instantaneous response of the heat flux to temperature gradients, implying an unphysical infinite speed of thermal propagation. This limitation becomes particularly significant in micro- and nano-scale applications or in materials where heat conduction exhibits delayed responses. By incorporating phase lag parameters into Eq. (5), our model accounts for the finite speed of thermal waves and captures the inherent delay in the heat flux response. This modification not only provides a more realistic description of heat transfer in systems with temperature-dependent properties but also accommodates the complex interactions introduced by rotational effects. As such, the non-standard Fourier law is essential for accurately modeling the dynamic behavior of heat conduction in our thermoelastic diffusion framework.

The relation between entropy, strain, temperature, and chemical potential is derived based on thermodynamic principles. We start from the first law of thermodynamics in a thermoelastic diffusion system.

Entropy-strain-temperature-chemical potential relation [20]:

$$\rho T_0 \dot{S} = l_1 T_0 \dot{T} + \gamma_1 T_0 \dot{e}_{kk} + d T_0 \dot{P} \tag{6}$$

Similarly, the diffusion equation is extended to include phase lags in mass flux and chemical potential gradients:

Mass concentration law:

$$-\eta_{i,i} + A_0 = \dot{C} \tag{7}$$

Equation of chemical potential [61]:

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} \frac{\partial^r}{\partial t^r}\right) \eta_i = -D \left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} \frac{\partial^r}{\partial t^r}\right) P_{,i} \tag{8}$$

This linear approximation is standard in thermoelastic diffusion theory [Biot (1956), Nowacki (1974)] and is valid for moderate temperature and concentration variations. While non-linear entropy effects can arise in extreme conditions (e.g., large deformations, phase transitions), the present study focuses on a small-deformation regime, where linear assumptions are widely accepted. Future work may extend this model to incorporate higher-order thermodynamic effects.

Mass concentration-chemical potential-strain-temperature relation [61]:

$$C = nP + \gamma_2 e_{kk} + dT \tag{9}$$

where

$$\lambda_0 = \lambda - \frac{\beta_2^2}{b}, \gamma_1 = \beta_1 + \frac{a}{b}\beta_2, \gamma_2 = \frac{\beta_2}{b}, \mathrm{d} = \frac{a}{b}, n = \frac{1}{b}, l_1 = \frac{\rho C_E}{T_0} + \frac{a^2}{b} \tag{10}$$

Here, the medium is rotating with angular velocity $\boldsymbol{\Omega} = \Omega \hat{\nu}$, where $\hat{\nu}$ is the unit vector along the axis of rotation and the equations of the motion (2) include two additional terms namely,

(1) The centripetal acceleration $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u})$ due to time-varying motion,

(2) The coriolis acceleration $2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}}$.

In the Eqs. (1)–(10), λ, μ-Lame's constants, ρ is the density, C_E is the specific heat at the constant strain, q_i are the components of heat flux vector **q**, F_i, u_i being the components of body force per unit mass and the components of the displacement vector **u**, respectively, K, S, P, and D related the thermal conductivity, entropy per unit mass, chemical potential and thermal diffusivity coefficient, respectively, $t_{ij}(=t_{ji})$, $e_{ij}(=e_{ji})$ $(i, j = 1, 2, 3)$ being the component of the stress tensor and the component of the straλin tensor, respectively, T$(= \theta - T_0)$ where T, Θ and T_0 concern to temperature increment, the absolute temperature of the medium and the reference temperature of the body respectively chosen such that, $|T/T_0| << 1$, $\beta_1 = (3\lambda + 2\mu)\alpha_t$, $\beta_2 = (3\lambda + 2\mu)\alpha_c$. α_t, α_c correspond to the coefficient of thermal linear expansion and diffusion expansion coefficient, respectively, δ_{ij} is Kronecker's delta, Q and A_0 are source of heat and source of chemical potential, respectively, a is the coefficient which describes the measure of thermoelastic diffusion, b is the coefficient of mass diffusion, thermal relaxation times with τ_q *and* $\tau_T \geq 0$ and diffusion relaxation times with τ_η and $\tau_P \geq 0$, δ^0 is constant taking the values 0 or 1, dot() denote time differentiation.

Temperature-dependent elastic parameters, thermal conductivity, and diffusion constant are taken as [62]

$$\lambda_0 = \lambda_{01} f(T) \lambda = \lambda_0^{'} f(T) \mu = \mu_0 f(T) K = K_0 f(T) \beta_1 = \beta_{10} f(T)$$

$$\beta_2 = \beta_{20} f(T) D = D_0 f(T) \tag{11}$$

where $f(T)$ is a non-dimensional function of temperature and the materials constants are where λ_{01}, $\lambda_0^{'}$, μ_0, K_0, β_{10}, β_{20} and D_0. $f(T)$ is a linear function of temperature for temperature-independent material qualities, i.e., $f(T) = (1 - \alpha^0 T)$, where $\alpha^0 [K^{-1}]$ is the empirical material constant. It produces non-linear governing equations that, if $|T/T_0| << 1$, can be linearized by approximating $f(T)$ so that $f(T) = (1 - \alpha^0 T)$.

With conditions in Eq. (11), Eqs. (1)–(9) reduce to

$$t_{ij} = 2\mu_0 f(T) e_{ij} + \delta_{ij} f(T) \left(\lambda_{01} e_{kk} - \gamma_{10} T - \gamma_{20} P\right) \tag{12}$$

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} \frac{\partial^r}{\partial t^r}\right) q_i = -K_0 f(T) \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} \frac{\partial^r}{\partial t^r}\right) T_{,i} \tag{13}$$

$$\rho T_0 \dot{S} = l_1 T_0 \dot{T} + \gamma_{10} f(T) T_0 \dot{e}_{kk} + d T_0 \dot{P} \tag{14}$$

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} \frac{\partial^r}{\partial t^r}\right) \eta_i = -D_0 f(T) \left(+ \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} \frac{\partial^r}{\partial t^r}\right) P_{,i} \tag{15}$$

$$C = nP + \gamma_{20} f(T) e_{kk} + dT \tag{16}$$

Eqs. (2), (4) and (7) with the aid of Eqs. (12)–(16) can be written as

$$(\lambda_{o1} + \mu_0) f(T) \nabla (\nabla.\mathbf{u}) - \mu_0 f(T) (\Delta \mathbf{u}) - \gamma_{10} f(T) \nabla \mathrm{T} - \gamma_{20} f(T) \nabla \mathrm{P} + \rho \nabla.\mathbf{F} - \rho \left(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u})\right)_i$$

$$- \rho \left(2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}}\right)_i = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \tag{17}$$

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} \frac{\partial^r}{\partial t^r}\right) \left(l_1 T_0 \dot{T} + \gamma_{10} f(T) T_0 \dot{e}_{kk} + d T_0 \dot{P} - Q\right) = K_0 f(T) \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} \frac{\partial^r}{\partial t^r}\right) \Delta \mathrm{T} \tag{18}$$

$$\left(\delta^0 + \sum\nolimits_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} \frac{\partial^r}{\partial t^r}\right) \left(n\dot{P} + \gamma_{20} f(T) \dot{e}_{kk} + d\dot{T} - A_0\right) = D_0 f(T) \left(1 + \sum\nolimits_{r=1}^{M^0} \frac{\tau_P^r}{r!} \frac{\partial^r}{\partial t^r}\right) \Delta P \tag{19}$$

Energy Theorem

Statement: The conservation of energy for a thermoelastic diffusion medium in a rotating frame can be derived by considering the time rate of change of total energy, including kinetic energy, strain energy, heat potential, and mass diffusion potential.

The energy theorem derived in this study describes the fundamental principle of energy conservation in a thermoelastic diffusion system. It accounts for contributions from mechanical work, thermal energy, and mass diffusion. The theorem states that the total energy of the system remains balanced when considering:

1. Kinetic Energy (KE: The energy associated with motion of the medium due to elastic deformations).
2. Strain Energy (SE: The potential energy stored in the material due to elastic deformation).
3. Heat Potential (HP: The energy associated with temperature variations and heat conduction).
4. Mass Diffusion Potential (MP: The energy contribution due to the movement of chemical species).

Conservation of energy for every time and every region V of ($\mathbb{R}^3$) is of the profile [14–17], mathematically, the energy balance equation is written as

$$\begin{aligned}&\frac{d}{dt}\left(K^*+W+P^*+M^*+d\int_V TPdV\right)=\tau_1^*\left[\int_A TT_{,i}n_i dA-\int_V T_{,i}T_{,i}dV\right]\\&+\tau_2^*\left[\int_A PP_{,i}n_i dA-\int_V P_{,i}P_{,i}dV\right]+\rho\int_V F_i u_i dV+\int_A h_i u_i dA+\frac{1}{T_0}\int_V QTdV+\int_V A_0 PdV\\&-\rho\int_V(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i\dot{u}_i dV-\rho\int_V(2\boldsymbol{\Omega}\times\dot{\boldsymbol{u}})_i\dot{u}_i dV\end{aligned}\tag{20}$$

Eq. (20) represents the conservation of energy in the thermoelastic diffusion system. The left-hand side denotes the time rate of change of total energy, including kinetic, strain, heat, and mass diffusion potential energies. The right-hand side accounts for mechanical work done by body forces, thermal and chemical contributions, as well as surface interactions due to stress and heat flux [28,31]. This theorem ensures that energy remains conserved under multi-phase-lag effects, rotation, and temperature-dependent properties. The formulation provides insights into how different physical fields interact in complex materials, with applications in semiconductor physics, thermal barrier coatings, and geophysical modeling. A deeper understanding of energy conservation in thermoelastic diffusion systems can aid in designing efficient energy management systems in electronic and optoelectronic devices.

Where K^*, W, P^* and M^* indicate the kinetic energy, the isothermal strain energy, heat potential and mass (diffusion) potential respectively and

$$\begin{aligned}&\tau_1^*=\frac{K_0 f(T)\tau_T^*}{T_0\tau_q^*},\tau_2^*=\frac{D_0 f(T)\tau_P^*}{\tau_\eta^*},\tau_q^*=\left(\delta^0+\sum\nolimits_{r=1}^{N^0}\frac{\tau_q^r}{r!}\frac{\partial^r}{\partial t^r}\right),\tau_T^*=\left(1+\sum\nolimits_{r=1}^{M^0}\frac{\tau_T^r}{r!}\frac{\partial^r}{\partial t^r}\right),\\&\tau_\eta^*=\left(\delta^0+\sum\nolimits_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}\frac{\partial^r}{\partial t^r}\right),\tau_P^*=\left(1+\sum\nolimits_{r=1}^{M^0}\frac{\tau_P^r}{r!}\frac{\partial^r}{\partial t^r}\right)\end{aligned}\tag{21}$$

Proof: let the Eqs. (12) and (2) require with the subsequent boundary conditions:

$$t_{ij}n_j=h_i\left(\boldsymbol{x},t\right),T\left(\boldsymbol{x},t\right)=\upsilon\left(\boldsymbol{x},t\right),P\left(\boldsymbol{x},t\right)=\mathit{\Omega}\left(\boldsymbol{x},t\right),\mathbf{x}\in A,\mathrm{t}>0\tag{22}$$

where n_j is the outward unit normal of A.

Multiplying Eq. (2) by $\dot{u}_i$ and integrating over V, we obtain

$$\int_V\left(t_{ij,j}+\rho F_i\right)\dot{u}_i dV=\rho\int_V\ddot{u}_i\dot{u}_i dV+\rho\int_V(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i\,\dot{u}_i dV+\rho\int_V(2\boldsymbol{\Omega}\times\dot{\boldsymbol{u}})_i\dot{u}_i dV\tag{23}$$

In light of Green's identity $t_{ij,j}\dot{u}_i = (t_{ij}\dot{u}_i)_{,j} - t_{ij}\dot{u}_{i,j}$ and Gauss's divergence theorem, Eq. (23) with the aid of Eq. (3) reduces to

$$\rho\int_V F_i\dot{u}_i dV + \int_A h_i\dot{u}_i dA = \rho\int_V \ddot{u}_i\dot{u}_i dV + \rho\int_V (\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i\,\dot{u}_i dV + \rho\int_V (2\boldsymbol{\Omega}\times\dot{\boldsymbol{u}})_i\,\dot{u}_i dV + \int_V t_{ij}\dot{e}_{ij} dV \tag{24}$$

Incorporating Eqs. (12) and (24), we obtain

$$\rho\int_V F_i\dot{u}_i dV + \int_A h_i\dot{u}_i dA = \rho\int_V \ddot{u}_i\dot{u}_i dV + \rho\int_V (\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i\,\dot{u}_i dV + \rho\int_V (2\boldsymbol{\Omega}\times\dot{\boldsymbol{u}})_i\,\dot{u}_i dV$$

$$+\int_V [2\mu_0 f(T)e_{ij} + \delta_{ij} f(T)\,(\lambda_{01}e_{kk} - \gamma_{10}T - \gamma_{20}P)]\dot{e}_{ij} dV \tag{25}$$

Introducing the following functions

$$W_0 = \mu_0\int_V e_{ij}e_{ij} dV + \frac{\lambda_{01}}{2}\int_V e_{kk}e_{kk} dV, \quad W = W_0 f\,(T) \tag{26}$$

is isothermal strain energy

$$K^* = \frac{\rho}{2}\int_V \dot{u}_i\dot{u}_i dV \tag{27}$$

Eq. (25) with the aid of (26) and (27) reduces to

$$\rho\int_V F_i\dot{u}_i dV + \int_A h_i\dot{u}_i dA = \frac{d}{dt}W + \frac{d}{dt}K^* + \rho\int_V (\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i\,\dot{u}_i dV$$

$$+\rho\int_V (2\boldsymbol{\Omega}\times\dot{\boldsymbol{u}})_i\dot{u}_i dV - \int_V \gamma_{10} f\,(T)\,T\dot{e}_{kk} dV - \int_V \gamma_{20} f(T)P\dot{e}_{kk} dV \tag{28}$$

The explanation of motion is contained in the first two integrals on the left-hand side of Eq. (28) (body force $= F_i$, surface traction $= t_i$), while the explanation of the remaining two integrals on the right-hand side of Eq. (28) includes sources of heat and mass diffusion that contribute to the body's heating.

Eq. (13) can be expressed as follows using (4) and (14) as a guide.

$$\left(\delta^0 + \sum_{r=1}^{N0}\frac{\tau_q^r}{r!}\frac{\partial^r}{\partial t^r}\right)(l_1 T_0\dot{T} + \gamma_{10} f\,(T)\,T_0\dot{e}_{kk} + dT_0\dot{P} - Q) = K_0 f(T)\left(1 + \sum_{r=1}^{M0}\frac{\tau_T^r}{r!}\frac{\partial^r}{\partial t^r}\right)T_{,ii} \tag{29}$$

Multiplying Eq. (29) with T and integrating over the region V, we determine

$$\tau_q^*\left[l_1 T_0\int_V \dot{T}T dV + \gamma_{10} f(T)T_0\int_V \dot{e}_{kk}T dV + dT_0\int_V \dot{P}T dV - \int_V QT dV\right] = K_0 f(T)\tau_T^*\int_V T_{,ii}T dV, \tag{30}$$

where

$$\tau_q^* = \left(\delta^0 + \sum_{r=1}^{N0}\frac{\tau_q^r}{r!}\frac{\partial^r}{\partial t^r}\right), \tau_T^* = \left(1 + \sum_{r=1}^{M0}\frac{\tau_T^r}{r!}\frac{\partial^r}{\partial t^r}\right) \tag{31}$$

In light of Green's identity $TT_{,ii} = \left(TT_{,i}\right)_{,i} - T_{,i}T_{,i}$ and dividing by T_0, Eq. (30) can be written as

$$\tau_q^*\left[l_1\int_V \dot{T}TdV + \gamma_{10}f(T)\int_V \dot{e}_{kk}TdV + d\int_V \dot{P}TdV - \frac{1}{T_0}\int_V QTdV\right] = \frac{K_0f(T)}{T_0}\tau_T^*\int_V\left(\left(TT_{,i}\right)_{,i} - T_{,i}T_{,i}\right)dV \tag{32}$$

Using the divergence theorem in Eq. (32), we obtain

$$\begin{aligned}&\tau_q^*\left[l_1\int_V \dot{T}TdV + \gamma_{10}f\left(T\right)\int_V \dot{e}_{kk}TdV + d\int_V \dot{P}TdV - \frac{1}{T_0}\int_V QTdV\right]\\ &= \frac{K_0f\left(T\right)}{T_0}\tau_T^*\int_A TT_{,i}n_idA - \frac{K_0f(T)}{T_0}\tau_T^*\int_V T_{,i}T_{,i}dV\end{aligned} \tag{33}$$

Introducing the following function

$$P^* = \frac{l_1}{2}\int_V T^2dV \tag{34}$$

Eq. (33) with the help of (34) takes the form

$$\begin{aligned}&\tau_q^*\left[\frac{dP^*}{dt} + \gamma_{10}f(T)\int_V \dot{e}_{kk}TdV + d\int_V \dot{P}TdV - \frac{1}{T_0}\int_V QTdV\right]\\ &= \frac{K_0f(T)}{T_0}\tau_T^*\int_A TT_{,i}n_idA - \frac{K_0f(T)}{T_0}\tau_T^*\int_V T_{,i}T_{,i}dV\end{aligned} \tag{35}$$

Eq. (16) along with Eqs. (7) and (15) take form

$$\left(\delta^0 + \sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}\frac{\partial^r}{\partial t^r}\right)\left(n\dot{P} + \gamma_{20}f(T)\dot{e}_{kk} + d\dot{T} - A_0\right) = D_0f(T)\left(1 + \sum_{r=1}^{M^0}\frac{\tau_P^r}{r!}\frac{\partial^r}{\partial t^r}\right)P_{,ii} \tag{36}$$

Multiplying Eq. (36) by P and integrating over the region V, we yield

$$\tau_\eta^*\left[n\int_V \dot{P}PdV + \gamma_{20}f(T)\int_V P\dot{e}_{kk}dV + d\int_V \dot{T}PdV - \int_V A_0PdV\right] = D_0f(T)\tau_P^*\int_V P_{,ii}PdV \tag{37}$$

where

$$\tau_\eta^* = \left(\delta^0 + \sum\nolimits_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}\frac{\partial^r}{\partial t^r}\right), \tau_P^* = \left(1 + \sum\nolimits_{r=1}^{M^0}\frac{\tau_P^r}{r!}\frac{\partial^r}{\partial t^r}\right) \tag{38}$$

In view of Green's identity $PP_{,ii} = \left(PP_{,i}\right)_{,i} - P_{,i}P_{,i}$and applying divergence theorem in Eq. (38) can be written as

$$\begin{aligned}&\tau_\eta^*[n\int_V \dot{P}PdV + \gamma_{20}f(T)\int_V P\dot{e}_{kk}dV + d\int_V \dot{T}PdV - \int_V A_0PdV] =\\ &D_0f(T)\tau_P^*\int_A PP_{,i}n_idA - D_0f(T)\tau_P^*\int_V P_{,i}P_{,i}dV\end{aligned} \tag{39}$$

Introduce the following function

$$M^{*} = \frac{n}{2}\int_{V} P^{2} dV \tag{40}$$

With the aid of (40), Eq. (39) can be written as

$$\tau_{\eta}^{*}\left[\frac{dM^{*}}{dt} + \gamma_{20} f(T)\int_{V} P\dot{e}_{kk} dV + d\int_{V}\dot{T} P dV - \int_{V} A_{0} P dV\right]$$
$$= D_{0} f(T)\tau_{P}^{*}\int_{A} P P_{,i} n_{i} dA - D_{0} f(T)\tau_{P}^{*}\int_{V} P_{,i} P_{,i} dV \tag{41}$$

Eq. (28) with the aid of (35) and (41) can be written as Eqs. (20) and (21).

Thus, the energy theorem is proved. □

Uniqueness Theorem

The uniqueness theorem demonstrates that the governing equations admit a single solution under homogeneous initial and boundary conditions. The proof utilizes the Laplace transform technique, which is particularly effective for solving and analyzing coupled thermoelastic diffusion systems. However, we acknowledge that in practical applications, boundary conditions may not always be homogeneous.

Theorem 1: *Let V be a domain in three-dimensional space* ($\mathbb{R}^{3}$) *enclosed by a piecewise smooth surface A in linear isotropic thermoelastic diffusion within a rotating frame of reference with temperature-dependent properties under a multi-phase-lag model. The functions* $u_{i}(\boldsymbol{x}, t)$, $T(\boldsymbol{x}, t)$, $P(\boldsymbol{x}, t)$ *of class* $C^{(m)}$ $(m \geq 2)$ *and* $t_{ij}(\boldsymbol{x}, t)$, $e_{ij}(\boldsymbol{x}, t)$ *of class* $C^{(l)}$ *for* $\boldsymbol{x} \in V + A$ *with coordinates* $\boldsymbol{x} = (x_{1}, x_{2}, x_{3})$ *at* $t \geq 0$ *that fulfil Eqs. (1)–(9) are equivalent, subject to the following border restrictions [29,30,57,58]:*

$$u_{i}(\boldsymbol{x}, t) = U_{i}(\boldsymbol{x}, t), T(\boldsymbol{x}, t) = \upsilon(\boldsymbol{x}, t), P(\boldsymbol{x}, t) = \Omega(\boldsymbol{x}, t); \boldsymbol{x} \in A, t > 0 \tag{42}$$

and the initial conditions at t = 0:

$$\mathrm{u}_{i}(\boldsymbol{x}, 0) = \mathrm{u}_{0i}(\mathbf{x}), \mathrm{T}(\boldsymbol{x}, 0) = T_{0}(\boldsymbol{x}), \mathrm{P}(\boldsymbol{x}, 0) = P_{0}(\boldsymbol{x});\ \text{for } \mathbf{x} \in \mathrm{V} \tag{43}$$

$$\frac{\partial^{k} u_{i}(\boldsymbol{x}, 0)}{\partial t^{k}} = u_{0i}^{(k)}(\boldsymbol{x}), k = 1, 2, 3, \ldots, \max\left\{N^{0}, M^{0}\right\}; \text{for x} \in V \tag{44}$$

$$\frac{\partial^{k} T(\boldsymbol{x}, 0)}{\partial t^{k}} = T_{0}^{(k)}(\mathbf{x}), k = 1, 2, 3, \ldots, \max\left\{N^{0}, M^{0}\right\}; \text{for x} \in V \tag{45}$$

$$\frac{\partial^{k} P(\boldsymbol{x}, 0)}{\partial t^{k}} = P_{0}^{(k)}(\mathbf{x}), k = 1, 2, 3, \ldots, \max\left\{N^{0}, M^{0}\right\}; \text{for x} \in V \tag{46}$$

where $\mathrm{u}_{0i}(\mathbf{x})$, $u_{0i}^{(k)}(\boldsymbol{x})$, $T_{0}(\boldsymbol{x})$, $T_{0}^{(k)}(\boldsymbol{x})$, $P_{0}(\boldsymbol{x})$ *and* $P_{0}^{(k)}(\boldsymbol{x})$ *are known functions.*

Let all field variables have a Laplace transformation, and let physical parameters account for the differences:

$$\lambda_{01} > 0, \mu_{0} > 0, \rho > 0, K_{0} > 0, \mathrm{C}_{\mathrm{E}} > 0, \mathrm{T}_{0} > 0, D_{0} > 0, \mathrm{d} > 0, n > 0, \tau_{T} > 0, \tau_{q} > 0, \tau_{P} > 0, \tau_{\eta} > 0$$

Solution: Let the two sets of solutions to Eqs. (2)–(4), (7) and (12)–(16) with homogeneous boundary and initial limitations be $u_i^{(1)}$, $T^{(1)}$, $P^{(1)}$, and $u_i^{(2)}$, $T^{(2)}$, $P^{(2)}$, ... Now let's take

$$\Pi^0 = \{u_i = u_i^{(1)} - u_i^{(2)}, \mathrm{T} = \mathrm{T}^{(1)} - \mathrm{T}^{(2)}, \mathrm{P} = \mathrm{P}^{(1)} - \mathrm{P}^{(2)}\} \tag{47}$$

Without body force, heat source, or mass diffusion source, the set Π^0 satisfies the governing Eqs. (2)–(4), (7) and (12)–(16). Additionally, functions u_i, T and P fulfill the homogeneous border and starting restrictions:

$$\mathrm{u}_i(\boldsymbol{x}, t) = 0, \mathrm{T}(\boldsymbol{x}, t) = 0, \mathrm{P}(\boldsymbol{x}, t) = 0; \mathbf{x} \in \mathrm{A}, \mathrm{t} > 0 \tag{48}$$

$$\mathrm{u}_i(\boldsymbol{x}, 0) = 0, \mathrm{T}(\boldsymbol{x}, 0) = 0, \mathrm{P}(\boldsymbol{x}, 0) = 0;\ \text{for } \mathbf{x} \in \mathrm{V} \tag{49}$$

$$\frac{\partial^k u_i(\boldsymbol{x}, 0)}{\partial t^k} = 0, k = 1, 2, 3, \ldots, \max\left\{N^0, M^0\right\}; \text{for x} \in V \tag{50}$$

$$\frac{\partial^k T(\boldsymbol{x}, 0)}{\partial t^k} = 0, k = 1, 2, 3, \ldots, \max\{N^0, M^0\};\ \text{for x} \in V \tag{51}$$

$$\frac{\partial^k P(\boldsymbol{x}, 0)}{\partial t^k} = 0, k = 1, 2, 3, \ldots, \max\{N^0, M^0\};\ \text{for x} \in V \tag{52}$$

While the Laplace transform simplifies the analysis by converting partial differential equations into algebraic equations, uniqueness can also be established through alternative techniques, such as energy methods or variational principles [Biot (1956), Nowacki (1974), Kumar and Gupta (2013b)]. For non-homogeneous boundary conditions, uniqueness still holds under specific compatibility constraints, ensuring well-posedness. Future work could extend this analysis to cases where boundary conditions include time-dependent or non-uniform external sources. Let the Laplace transform be defined as

$$\bar{f}(\boldsymbol{x}, \mathrm{s}) = \mathrm{L}(\mathrm{f}(\boldsymbol{x}, t)) = \int_0^\infty \mathrm{f}(\boldsymbol{x}, t)\, e^{-st} \mathrm{d}t; \mathrm{s} > 0 \tag{53}$$

For additional simplification, apply the Laplace transform to Eqs. (2)–(4), (7), (12)–(16), (48) and (49) after eliminating the body forces, heat source, and chemical potential source, as well as the bar.

$$t_{ij} = 2\mu_0 f(T) e_{ij} + \delta_{ij} f(T)(\lambda_{01} e_{kk} - \gamma_{10} T - \gamma_{20} P) \tag{54}$$

$$t_{ij,j} = \rho s^2 u_i \tag{55}$$

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{56}$$

$$q_{i,i} = \rho T_0 s S \tag{57}$$

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right) q_i = -K_0 f(T) \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right) T_i \tag{58}$$

$$\rho T_0 s S = l_1 T_0 s T + \gamma_{10} f(T) T_0 s e_{kk} + d T_0 s P \tag{59}$$

$$-\eta_{i,i} = sC \tag{60}$$

$$\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}s^r\right)\eta_i=-D_0f(T)\left(1+\sum_{r=1}^{M^0}\frac{\tau_P^r}{r!}s^r\right)P_{,i} \tag{61}$$

$$C=nP+\gamma_{20}f(T)e_{kk}+dT \tag{62}$$

$$u_i\,(\boldsymbol{x},s)=0,\,T\,(\boldsymbol{x},s)=0,\,P\,(\boldsymbol{x},s)=0;\,\boldsymbol{x}\in A \tag{63}$$

$$\mathrm{u}_i\,(\boldsymbol{x},0)=0,\,\mathrm{T}\,(\boldsymbol{x},0)=0,\,\mathrm{P}\,(\boldsymbol{x},0)=0;\ \text{for }\mathbf{x}\in\mathrm{V} \tag{64}$$

Consider the integral

$$\int_V t_{ij}e_{ij}dV=\int_V t_{ij}u_{i,j}dV=\int_V\left(t_{ij}u_i\right)_{,j}dV-\int_V t_{ij,j}u_idV \tag{65}$$

With the help of the divergence theorem and owing to Eq. (63), yield

$$\int_V\left(t_{ij}u_i\right)_{,j}dV=\int_A u_it_{ij}n_jdA=0 \tag{66}$$

Eq. (65) with the aid of (66) takes the form

$$\int_V t_{ij}e_{ij}dV+\int_V t_{ij,j}u_idV=0 \tag{67}$$

Eq. (67) with the aid of (54) and (55) can be expressed in the form

$$\int_V[2\mu_0f(T)e_{ij}+\delta_{ij}f(T)\,(\lambda_{01}e_{kk}-\gamma_{10}T-\gamma_{20}P)]e_{ij}\mathrm{dV}\;+\int_V\rho s^2u_i^2dV=0 \tag{68}$$

To determine, $\gamma_{10}f(T)\int_V Te_{kk}dV$, following Biot [61], let us introduce a vector function ϕ_i associated with entropy through the alliance

$$q_i=T_0\dot{\phi}_i,\,\rho S=-\phi_{i,i} \tag{69}$$

Invoking Eq. (69) in Eqs. (58) and (59) respectively and assuming that $\phi_i(\boldsymbol{x},0)=\dot{\phi}_i(\boldsymbol{x},0)=0$, we obtain, respectively

$$\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_q^r}{r!}s^r\right)sT_0\phi_i=-K_0f(T)\left(1+\sum_{r=1}^{M^0}\frac{\tau_T^r}{r!}s^r\right)T_{,i} \tag{70}$$

$$-\,T_0s\phi_{i,i}=l_1T_0sT+\gamma_{10}f(T)T_0se_{kk}+dT_0sP \tag{71}$$

Multiply both sides of Eq. (70) by ϕ_i and integrating over the region V of the body, we obtain

$$\int_V\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_q^r}{r!}s^r\right)sT_0\phi_i^2dV=-K_0f(T)\int_V\left(1+\sum_{r=1}^{M^0}\frac{\tau_T^r}{r!}s^r\right)T_{,i}\phi_idV \tag{72}$$

With the help of divergence theorem and owing to Eq. (63), Eq. (72) can be written as

$$\int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right) sT_0\phi_i^2 dV = K_0 f(T) \int_V \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right) T\phi_{i,i} dV \tag{73}$$

Incorporating Eq. (71) in Eq. (73), we obtain

$$-\gamma_{10} f(T) \int_V Te_{kk} dV = \frac{1}{K_0 f(T)\left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right)} \int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right) sT_0\phi_i^2 dV + \int_V l_1 T^2 dV + \int_V PdTdV \tag{74}$$

To determine $\gamma_{20} f(T) \int_V Pe_{kk} dV$, let us introduce the vector function N_i defined as

$$\eta_i = \dot{N}_i,\ C = -N_{i,i} \tag{75}$$

Substituting the value of η_i and C from Eq. (75) in Eqs. (61) and (62), respectively, and assuming that $N_i(\boldsymbol{x}, 0) = \dot{N}_i(\boldsymbol{x}, 0) = 0$, we obtain, respectively

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} s^r\right) sN_{,i} = -D_0 f(T) \left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} s^r\right) P_{,i} \tag{76}$$

$$-N_{i,i} = nP + \gamma_{20} f(T) e_{kk} + dT \tag{77}$$

Multiplying both sides of (76) by N_i and integrating over the region of the body, we obtain

$$\int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} s^r\right) sN_i^2 dV = -D_0 f(T) \int_V \left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} s^r\right) P_{,i} N_i dV \tag{78}$$

With the help of the divergence theorem and owing to Eq. (63), Eq. (78) can be written as

$$\int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} s^r\right) sN_i^2 dV = D_0 f(T) \int_V \left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} s^r\right) PN_{i,i} dV \tag{79}$$

Using Eq. (77) in Eq. (79), we obtain

$$-\gamma_{20} f(T) \int_V Pe_{kk} dV = \frac{1}{D_0 f(T)\left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} s^r\right)} \int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} s^r\right) sN_i^2 dV + \int_V nP^2 dV + \int_V PdTdV \tag{80}$$

Using the value of $\gamma_{10} f(T) \int_V T e_{kk} dV$ from Eq. (74) and $\gamma_{20} f(T) \int_V P e_{kk} dV$ from (80) in Eq. (68), we obtain

$$\int_V 2\mu_0 f(T)\, e_{ij} e_{ij} dV + \int_V \lambda_{01} f(T)\, e_{kk} e_{kk} dV + \int_V \rho s^2 u_i^2 dV + \frac{1}{K_0 f(T) \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right)} \int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right) s T_0 \phi_i^2 dV + \int_V l_1 T^2 dV + \frac{1}{D_0 f(T) \left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} s^r\right)} \int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} s^r\right) s N_i^2 dV + \int_V n P^2 dV + 2 \int_V P d T dV = 0 \tag{81}$$

It is evident that $\int_V (l_1 T^2 + nP^2 + 2PdT) dV$ appearing in the expression (81) is always positive because of new thermodynamics law [Nowacki (1974a)]

$$0 < d^2 < l_1 n \tag{82}$$

Due to the homogeneous beginning circumstances, the (81) expression disappears at t = 0 and must always be non-positive for t > 0.

Given that the integrand function in Eq. (81) is a sum of squares and the material parameters are positive, we can thus deduce that

$$u_i = T = e_{ij} = t_{ij} = P = 0 \tag{83}$$

According to Learch's theorem [61], the inverse Laplace transform of each is unique, meaning that the Laplace transforms of the difference function (47) are zeros.

$$u_i^{(1)} = u_i^{(2)}, T^{(1)} = T^{(2)}, P^{(1)} = P^{(2)} \tag{84}$$

Hence, the uniqueness theorem is proved owing to initial and boundary restrictions. The importance of uniqueness lies in ensuring that the physical model is well-posed and that solutions are stable under small perturbations. This guarantees that predictions made using this model are physically meaningful and can be reliably applied in fields such as material science, semiconductor physics, and geophysics.

Reciprocity Theorem

The reciprocity theorem establishes a fundamental relationship between displacement, temperature, and chemical potential in thermoelastic diffusion under varying boundary conditions. Mathematically, it ensures that if an external force, heat source, or chemical concentration is applied at one point, the response at another point in the system is symmetrically related.

Physical Interpretation:

Mechanical Reciprocity: If a force is applied at point A, leading to a displacement at point BBB, then the same force applied at B would produce an equal displacement at A.

Thermal Reciprocity: If a heat source is introduced at A, influencing the temperature at B, then reversing the source and response would yield the same effect.

Diffusion Reciprocity: A change in chemical potential at one boundary affects mass transport at another, and *vice versa*, which is critical in semiconductor physics and porous media.

Let V be a bounded domain in $\mathbb{R}^3$ with a piecewise smooth boundary A. Consider a multi-phase-lag thermoelastic diffusion system in a rotating frame, where the displacement u_i, temperature T, and chemical potential P satisfy the governing Eqs. (2), (12)–(19) (are continuous and have continuous derivatives up to the second order). When considering $\mathbf{x} \in V + A$, $t > 0$, we take into account that the stress t_{ij} and the strains e_{ij} are continuous and have first derivatives [30,57].

Eq. (12) is obtained by incorporating Eq. (2).

$$\left[2\mu_0 f(T)e_{ij} + \delta_{ij} f(T)\left(\lambda_{01}e_{kk} - \gamma_{10}T - \gamma_{20}P\right)\right]_{,j} + \rho F_i = \rho\ddot{u}_i + \rho(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i + \rho(2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})_i \tag{85}$$

We assume that the system of Eqs. (18), (19) and (85) are assists with the boundary condition given by (22) and homogeneous initial conditions (49)–(52).

To obtain the dynamic reciprocity relationship for TDMR, one must consider the effects of the body forces $F_i(\boldsymbol{x}, t)$, boundary condition (22), homogeneous initial condition (49)–(52), surface traction $h_i(\boldsymbol{x}, t)$, chemical potential $\Omega(\boldsymbol{x}, t)$, and surface heating to $\upsilon(\boldsymbol{x}, t)$.

Exploring Laplace transform defined by Eq. (53) on Eq. (2), (18), (19), (85) and (22)

$$t_{ij,j} + \rho F_i = \rho s^2 u_i + \rho(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i + \rho(2\boldsymbol{\Omega} \times s\boldsymbol{u})_i \tag{86}$$

$$\left[2\mu_0 f(T)e_{ij} + \delta_{ij} f(T)\left(\lambda_{01}e_{kk} - \gamma_{10}T - \gamma_{20}P\right)\right]_{,j} + \rho F_i = \rho s^2 u_i + \rho(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i + \rho(2\boldsymbol{\Omega} \times s\boldsymbol{u})_i \tag{87}$$

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right)\left(l_1 T_0 sT + \gamma_{10} f(T)\, T_0 s e_{kk} + dT_0 sP - Q\right) = K_0 f(T)\left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right) T_{,ii} \tag{88}$$

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} s^r\right)\left(nsP + \gamma_{20} f(T)\, s e_{kk} + dTs - A_0\right) = D_0 f(T)\left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} s^r\right) P_{,ii} \tag{89}$$

$$t_{ij}n_j = h_i(\boldsymbol{x}, s),\ T(\boldsymbol{x}, s) = \upsilon(\boldsymbol{x}, s),\ P(\boldsymbol{x}, s) = \Omega(\boldsymbol{x}, s),\ \boldsymbol{x} \in A \tag{90}$$

We now look at two issues with independent specifications for the applied body force, surface temperature, and chemical potential. Let the superscript enclosed in parenthesis represent the variable that appears in these two problems. So, we start with the first problem: $u_i^{(1)}$, $e_{ij}^{(1)}$, $t_{ij}^{(1)}$, $T^{(1)}$, $P^{(1)}$, . . . In contrast, the second problem involves the following: $u_i^{(2)}$, $e_{ij}^{(2)}$, $t_{ij}^{(2)}$, $T^{(2)}$, $P^{(2)}$, . . . The system of Eqs. (64) and (86)–(90) are satisfied by every collection of variables.

Let us consider the integral

$$\int_V t_{ij}^{(1)} e_{ij}^{(2)} dV = \int_V t_{ij}^{(1)} u_{i,j}^{(2)} dV = \int_A t_{ij}^{(1)} u_i^{(2)} dA - \int_V t_{ij,j}^{(1)} u_i^{(2)} dV \tag{91}$$

Using the divergence theorem, the symmetry of t_{ij} and incorporating (86) and (90) in Eq. (91), determine

$$\int_V t_{ij}^{(1)} e_{ij}^{(2)} dV = \int_A h_i^{(1)} u_i^{(2)} dA - \rho\int_V s^2 u_i^{(1)} u_i^{(2)} dV + \rho\int_V F_i^{(1)} u_i^{(2)} dV - \rho\int_V \left[(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i^{(1)} + (2\boldsymbol{\Omega} \times s\boldsymbol{u})_i^{(1)}\right] u_i^{(2)} \mathrm{dV} \tag{92}$$

According to Eq. (92), the integral $\int_V t_{ij}^{(2)} e_{ij}^{(1)} dV$ can be written as

$$\int_V t_{ij}^{(2)} e_{ij}^{(1)} dV = \int_A h_i^{(2)} u_i^{(1)} dA - \rho \int_V s^2 u_i^{(2)} u_i^{(1)} dV + \rho \int_V F_i^{(2)} u_i^{(1)} dV - \rho \int_V [(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i^{(2)} + (2\boldsymbol{\Omega} \times s\boldsymbol{u})_i^{(2)}] u_i^{(1)} \mathrm{dV} \tag{93}$$

Subtracting Eqs. (92) and (93), we obtain

$$= \int_V \left(t_{ij}^{(1)} e_{ij}^{(2)} - t_{ij}^{(2)} e_{ij}^{(1)}\right) dV = \int_A \left(h_i^{(1)} u_i^{(2)} - h_i^{(2)} u_i^{(1)}\right) dA + \rho \int_V \left(F_i^{(1)} u_i^{(2)} - F_i^{(2)} u_i^{(1)}\right) dV$$
$$- \rho \int_V [(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i^{(1)} u_i^{(2)} - (\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i^{(2)} u_i^{(1)}] dV - \rho \int_V [(2\boldsymbol{\Omega} \times s\boldsymbol{u})_i^{(1)} u_i^{(2)} - (2\boldsymbol{\Omega} \times s\boldsymbol{u})_i^{(2)} u_i^{(1)}] \mathrm{dV} \tag{94}$$

Eq. (54) is multiplied by $e_{ij}^{(2)}$ and $e_{ij}^{(1)}$ for the first and second problem, respectively, subtracting and integrating over the region V, we obtain

$$\int_V \left(t_{ij}^{(1)} e_{ij}^{(2)} - t_{ij}^{(2)} e_{ij}^{(1)}\right) dV = \int_V [\gamma_{10} f(T) \left(T^{(2)} e^{(1)} - T^{(1)} e^{(2)}\right) - \gamma_{20} f(T) \left(P^{(1)} e^{(2)} - P^{(2)} e^{(1)}\right)] \tag{95}$$

where $e_{ij}\delta_{ij} = e_{kk} = e$

Using Eq. (94) in Eq. (95) determine

$$\int_A \left(h_i^{(1)} u_i^{(2)} - h_i^{(2)} u_i^{(1)}\right) dA + \rho \int_V \left(F_i^{(1)} u_i^{(2)} - F_i^{(2)} u_i^{(1)}\right) dV - \rho \int_V^{;} [(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i^{(1)} u_i^{(2)}$$
$$- (\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i^{(2)} u_i^{(1)}] dV - \rho \int_V^{;} \left[(2\boldsymbol{\Omega} \times s\boldsymbol{u})_i^{(1)} u_i^{(2)} - (2\boldsymbol{\Omega} \times s\boldsymbol{u})_i^{(2)} u_i^{(1)}\right] \mathrm{dV}$$
$$= \int_V \left[\gamma_{10} f(T) \left(T^{(2)} e^{(1)} - T^{(1)} e^{(2)}\right) - \gamma_{20} f(T) \left(P^{(1)} e^{(2)} - P^{(2)} e^{(1)}\right)\right] dV \tag{96}$$

To derive the second part, multiply (88) by T$^{(2)}$ and T$^{(1)}$for the first and secthond problem, respectively, subtract and integrate over the region V

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right) [\gamma_{10} f(T) s T_0 \int_V (e^{(1)} T^{(2)} - e^{(2)} T^{(1)}) dV + d T_0 s \int_V (P^{(1)} T^{(2)} - P^{(2)} T^{(1)}) dV$$
$$- \int_V \left(Q^{(1)} T^{(2)} - Q^{(2)} T^{(1)}\right) dV] = K_0 f(T) \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right) \int_V \left(T_{,ii}^{(1)} T^{(2)} - T_{,ii}^{(2)} T^{(1)}\right) dV \tag{97}$$

With the help of the divergence theorem and Eq. (90), Eq. (97) takes the form

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right) [\gamma_{10} f(T) s T_0 \int_V (e^{(1)} T^{(2)} - e^{(2)} T^{(1)}) dV + d T_0 s \int_V (P^{(1)} T^{(2)} - P^{(2)} T^{(1)}) dV$$
$$- \int_V (Q^{(1)} T^{(2)} - Q^{(2)} T^{(1)}) dV] = K_0 f(T) \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right) \int_A (T_{,i}^{(1)} \upsilon^{(2)} - T_{,i}^{(2)} \upsilon^{(1)}) dA \tag{98}$$

Eq. (98) established the second part of the reciprocity theorem which involves the thermal causes of the motion υ: heating of the surface A.

To derive the third part, multiply (89) by $P^{(2)}$ and $P^{(1)}$ to the first and second problem, respectively, then integrate over the volume V.

$$\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}s^r\right)[\gamma_{20}f(T)\text{s}\int_V\left(e^{(1)}P^{(2)}-e^{(2)}P^{(1)}\right)dV+ds\int_V\left(T^{(1)}P^{(2)}-T^{(2)}P^{(1)}\right)dV$$
$$-\int_V(A_0^{(1)}P^{(2)}-A_0^{(2)}P^{(1)})dV]=D_0f(T)\left(1+\sum_{r=1}^{M^0}\frac{\tau_P^r}{r!}s^r\right)\int_V\left(P_{,ii}^{(1)}P^{(2)}-P_{,ii}^{(2)}P^{(1)}\right)dV \tag{99}$$

With the help of the divergence theorem and Eq. (90), Eq. (99) takes the form

$$\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}s^r\right)[\gamma_{20}f(T)s\int_V\left(e^{(1)}P^{(2)}-e^{(2)}P^{(1)}\right)dV+sd\int_V\left(T^{(1)}P^{(2)}-T^{(2)}P^{(1)}\right)dV$$
$$-\int_V\left(A_0^{(1)}P^{(2)}-A_0^{(2)}P^{(1)}\right)dV]=D_0f(T)\left(1+\sum_{r=1}^{M^0}\frac{\tau_P^r}{r!}s^r\right)\int_A\left(P_{,i}^{(1)}\Omega^{(2)}-P_{,i}^{(2)}\Omega^{(1)}\right)dA \tag{100}$$

Eq. (100) establishes the third part of the reciprocity theorem which contains the chemical potential causes of motion.

Using the integral $\int_V(e^{(1)}T^{(2)}-e^{(2)}T^{(1)})\,dV$ from Eq. (98) and $\int_V\left(e^{(1)}P^{(2)}-e^{(2)}P^{(1)}\right)dV$ from Eq. (100) in Eq. (96), we obtain

$$K_0f\,(T)\left(1+\sum_{r=1}^{M^0}\frac{\tau_T^r}{r!}s^r\right)R_2\int_A\left(T_{,i}^{(1)}\upsilon^{(2)}-T_{,i}^{(2)}\upsilon^{(1)}\right)dA+T_0R_1D_0f\,(T)\left(1+\sum_{r=1}^{M^0}\frac{\tau_P^r}{r!}s^r\right)$$
$$\int_A\left(P_{,i}^{(1)}\Omega^{(2)}-P_{,i}^{(2)}\Omega^{(1)}\right)dA-R_1sT_0R_2v\int_A\left(h_i^{(1)}u_i^{(2)}-h_i^{(2)}u_i^{(1)}\right)dA-\rho sT_0R_1R_2$$
$$\int_V\left(F_i^{(1)}u_i^{(2)}-F_i^{(2)}u_i^{(1)}\right)dV+\rho sT_0R_1R_2\int_V[(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i^{(1)}u_i^{(2)}-(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i^{(2)}u_i^{(1)}]dV$$
$$+\rho sT_0R_1R_2\int_V[(2\boldsymbol{\Omega}\times s\boldsymbol{u})_i^{(1)}u_i^{(2)}-(2\boldsymbol{\Omega}\times s\boldsymbol{u})_i^{(2)}u_i^{(1)}]\text{dV}$$
$$+R_1R_2\int_V(Q^{(1)}T^{(2)}-Q^{(2)}T^{(1)})dV+T_0R_1R_2\int_V\left(A_0^{(1)}P^{(2)}-A_0^{(2)}P^{(1)}\right)dV=0 \tag{101}$$

where $\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_q^r}{r!}s^r\right)=R_1,\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}s^r\right)=R_2$

Eq. (101) represents the reciprocity theorem (general form) in the new domain.

For the implementation of the inverse Laplace transform on (96), (98), (100) and (101), the following convolution theorem and symbolic notations are used

$$L^{-1}\{F\,(s)\,G\,(s)\}=\int_0^t f\,(t-\zeta)\,g\,(\zeta)\,d\zeta=\int_0^t g\,(t-\zeta)\,f\,(\zeta)\,d\zeta \tag{102}$$

$$\Lambda_1\,(f\,(\boldsymbol{x},\zeta))=\left(\delta^0f\,(\boldsymbol{x},\zeta)+\sum_{r=1}^{N^0}\frac{\tau_q^r}{r!}\frac{\partial^rf\,(\boldsymbol{x},\zeta)}{\partial\zeta^r}\right) \tag{103}$$

$$\Lambda_2\,(f\,(\boldsymbol{x},\zeta))=\left(\delta^0f\,(\boldsymbol{x},\zeta)+\sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}\frac{\partial^rf\,(\boldsymbol{x},\zeta)}{\partial\zeta^r}\right) \tag{104}$$

$$\Lambda_3\left(f\left(\boldsymbol{x},\zeta\right)\right)=\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_q^r}{r!}\frac{\partial^r}{\partial\zeta^r}\right)\left(\delta^0 f\left(\boldsymbol{x},\zeta\right)+\sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}\frac{\partial^r f\left(\boldsymbol{x},\zeta\right)}{\partial\zeta^r}\right)\tag{105}$$

Thus, we obtain the first, second, third and general form of the reciprocity theorem in the resulting form.

$$\int_A\int_0^t h_i^{(1)}\left(\boldsymbol{x},t-\zeta\right)u_i^{(2)}\left(\boldsymbol{x},\zeta\right)d\zeta\,dA+\rho\int_V\int_0^t F_i^{(1)}\left(\boldsymbol{x},t-\zeta\right)u_i^{(2)}\left(\boldsymbol{x},\zeta\right)d\zeta\,dV$$

$$+\gamma_{10}f\left(T\right)\int_V\int_0^t T^{(1)}\left(\boldsymbol{x},t-\zeta\right)e^{(2)}\left(\boldsymbol{x},\zeta\right)d\zeta\,dV+\gamma_{20}f\left(T\right)\int_V\int_0^t P^{(1)}\left(\boldsymbol{x},t-\zeta\right)e^{(2)}\left(\boldsymbol{x},\zeta\right)d\zeta\,dV$$

$$-\rho\int_V\int_0^t\left(\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\boldsymbol{u}\right)\right)_i^{(1)}\left(\boldsymbol{x},t-\zeta\right)u_i^{(2)}\left(\boldsymbol{x},\zeta\right)d\zeta\,dV-\rho\int_V\int_0^t\left(2\boldsymbol{\Omega}\times\frac{\partial}{\partial\zeta}\boldsymbol{u}\right)_i^{(1)}$$

$$\left(\boldsymbol{x},t-\zeta\right)u_i^{(2)}\left(\boldsymbol{x},\zeta\right)d\zeta=S_{21}^{12}\tag{106}$$

$$\gamma_{10}f(T)T_0\int_V\int_0^t e^{(1)}\left(\boldsymbol{x},t-\zeta\right)\frac{\partial}{\partial\zeta}\Lambda_1\left(T^{(2)}\left(\boldsymbol{x},\zeta\right)\right)d\zeta\,dV+dT_0\int_V\int_0^t-P^{(1)}\left(\boldsymbol{x},t-\zeta\right)\frac{\partial}{\partial\zeta}\Lambda_1\left(T^{(2)}\left(\boldsymbol{x},\zeta\right)\right)$$

$$d\zeta\,dV-\int_V\int_0^t Q^{(1)}\left(\boldsymbol{x},t-\zeta\right)\Lambda_1\left(T^{(2)}\left(\boldsymbol{x},\zeta\right)\right)d\zeta\,dV-K_0f(T)\int_A\int_0^t T_{,i}^{(1)}$$

$$\left(\boldsymbol{x},t-\zeta\right)\upsilon^{(2)}\left(\boldsymbol{x},\zeta\right)d\zeta\,dA-K_0f(T)\sum_{r=1}^{M^0}\frac{\tau_T^r}{r!}\int_A\int_0^t T_{,i}^{(1)}\left(\boldsymbol{x},t-\zeta\right)\frac{\partial^r}{\partial\zeta^r}\upsilon^{(2)}\left(\boldsymbol{x},\zeta\right)d\zeta\,dA=S_{21}^{12}\tag{107}$$

$$D_0f(T)\int_A\int_0^t P_{,i}^{(1)}\left(\boldsymbol{x},t-\zeta\right)\mathit{\Omega}^{(2)}\left(\boldsymbol{x},\zeta\right)d\zeta\,dA+D_0f(T)\sum_{r=1}^{M^0}\frac{\tau_P^r}{r!}\int_A\int_0^t P_{,i}^{(1)}$$

$$\left(\boldsymbol{x},t-\zeta\right)\frac{\partial^r}{\partial\zeta^r}\mathit{\Omega}^{(2)}\left(\boldsymbol{x},\zeta\right)d\zeta\,dA-\gamma_{20}f(T)\int_V\int_0^t e^{(1)}\left(\boldsymbol{x},t-\zeta\right)\frac{\partial}{\partial\zeta}\Lambda_2\left(P^{(2)}\left(\boldsymbol{x},\zeta\right)\right)d\zeta\,dV-\mathrm{d}\int_V\int_0^t T^{(1)}$$

$$\left(\boldsymbol{x},t-\zeta\right)\frac{\partial}{\partial\zeta}\Lambda_2\left(P^{(2)}\left(\boldsymbol{x},\zeta\right)\right)d\zeta\,dV+\int_V\int_0^t A_0^{(1)}\left(\boldsymbol{x},t-\zeta\right)\Lambda_2\left(P^{(2)}\left(\boldsymbol{x},\zeta\right)\right)d\zeta\,dV=S_{21}^{12}\tag{108}$$

$$K_0f\left(T\right)\int_A\int_0^t T_{,i}^{(1)}\left(\boldsymbol{x},t-\zeta\right)\Lambda_2\left(\upsilon^{(2)}\left(\boldsymbol{x},\zeta\right)\right)d\zeta\,dA+K_0f\left(T\right)\sum_{r=1}^{M^0}\frac{\tau_T^r}{r!}\int_A\int_0^t T_{,i}^{(1)}$$

$$\left(\boldsymbol{x},t-\zeta\right)\frac{\partial^r}{\partial\zeta^r}\Lambda_2\left(\upsilon^{(2)}\left(\boldsymbol{x},\zeta\right)\right)d\zeta\,dA+D_0f\left(T\right)T_0\int_A\int_0^t P_{,i}^{(1)}$$

$$\left(\boldsymbol{x},t-\zeta\right)\Lambda_1\left(\mathit{\Omega}^{(2)}\left(\boldsymbol{x},\zeta\right)\right)d\zeta\,dA+T_0D_0f\left(T\right)\sum_{r=1}^{M^0}\frac{\tau_P^r}{r!}\int_A\int_0^t P_{,i}^{(1)}$$

$$\left(\boldsymbol{x},t-\zeta\right)\frac{\partial^r}{\partial\zeta^r}\Lambda_1\left(\mathit{\Omega}^{(2)}\left(\boldsymbol{x},\zeta\right)\right)d\zeta\,dA-\rho T_0\int_V\int_0^t F_i^{(1)}\left(\boldsymbol{x},t-\zeta\right)\frac{\partial}{\partial\zeta}\Lambda_3\left(u_i^{(2)}\left(\boldsymbol{x},\zeta\right)\right)d\zeta\,dV-$$

$$T_0\int_A\int_0^t h_i^{(1)}\left(\boldsymbol{x},t-\zeta\right)\frac{\partial}{\partial\zeta}\Lambda_3\left(u_i^{(2)}\left(\boldsymbol{x},\zeta\right)\right)d\zeta\,dA$$

$$+\rho T_0\int_V\int_0^t (\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i^{(1)}(\boldsymbol{x},t-\zeta)\frac{\partial}{\partial\zeta}\Lambda_3\left(u_i^{(2)}(\boldsymbol{x},\zeta)\right)d\zeta dV+\rho T_0\int_V\int_0^t\left(2\boldsymbol{\Omega}\times\frac{\partial}{\partial\zeta}\boldsymbol{u}\right)_i^{(1)}$$

$$(\boldsymbol{x},t-\zeta)\frac{\partial}{\partial\zeta}\Lambda_3\left(u_i^{(2)}(\boldsymbol{x},\zeta)\right)d\zeta dV+\int_V\int_0^t Q^{(1)}(\boldsymbol{x},t-\zeta)\Lambda_3\left(T^{(2)}(\boldsymbol{x},\zeta)\right)d\zeta dV$$

$$+T_0\int_V\int_0^t A_0^{(1)}(\boldsymbol{x},t-\zeta)\Lambda_3\left(P^{(2)}(\boldsymbol{x},\zeta)\right)d\zeta dV=S_{21}^{12}\tag{109}$$

Here S_{21}^{12} indicates the same expression on the left-hand side except that the superscripts (1) and (2) are interchanged. Hence the reciprocity theorem is proved.

Real-world applications of the reciprocity theorem are:

- In semiconductors, this theorem helps in understanding photo-thermoelastic wave interactions, where heat and mechanical stresses affect electronic transport.
- In geomechanics, it explains how thermal expansion in rocks due to underground heat sources influences stress distribution.
- In thermal barrier coatings, reciprocity principles guide the design of materials to minimize heat-induced stress and diffusion.

Thus, the reciprocity theorem provides a powerful tool for predicting system behavior under different applied conditions and ensures energy and response symmetry in complex thermoelastic materials

Variational Criterion

The criterion of virtual work with variations of displacement for deformable elastic body with thermal and chemical potential is given as [62]

$$\int_V\rho\left(F_i-\ddot{u}_i+(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i+(2\boldsymbol{\Omega}\times\dot{\boldsymbol{u}})_i\right)\delta u_i dV+\int_A h_i\delta u_i dA=\int_V t_{ij}\delta u_{i,j}dV\tag{110}$$

While the right side of Eq. (110) relates to the virtual work of internal forces, the left side corresponds to the virtual work of body forces F_i, inertial forces $\rho\ddot{u}_i$, and surface force $h_i=t_{ji}n_j$. We indicate the outward unit normal of A by n_i. Eq. (110) can be rewritten as follows using the definition of the strain tensor and the symmetry of the stress tensor:

Substitute the value of t_{ij} from (1) in Eq. (110), we obtain

$$\int_V\rho\left(F_i-\ddot{u}_i+(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i+(2\boldsymbol{\Omega}\times\dot{\boldsymbol{u}})_i\right)\delta u_i dV+\int_A h_i\delta u_i dA$$

$$=\delta W-\gamma_{10}f(T)\int_V \mathrm{T}\delta e_{kk}dV-\gamma_{20}f(T)\int_V P\delta e_{kk}dV\tag{111}$$

where

$$W_0=\mu_0 f(T)\int_V e_{ij}e_{ij}dV+\frac{\lambda_{01}}{2}\int_V e_{kk}e_{kk}dV,\mathrm{W}=W_0 f(T)$$

is the strain density function.

Introducing a vector function ϕ_i (following Biot [61]) connected with entropy and heat source term Q_1 through relations:

$$q_i = T_0\dot{\phi}_i,\ Q = T_0\dot{Q}_1 \tag{112}$$

Then, the Eqs. (4) and (13) with the aid of (14) and (112) reduce to

$$\delta\phi_{i,i} = -l_1\delta T - \gamma_{10}f\left(T\right)\delta e_{kk} - \mathrm{d}\delta\mathrm{P} + \delta Q_1 \tag{113}$$

$$\frac{T_0}{K_0 f(T)}J_i + T_{,i} = 0 \tag{114}$$

where

$$J_i = \left(\delta^0 + \sum_{r=1}^{N0}\frac{\tau_q^r}{r!}\frac{\partial^r}{\partial t^r}\right)\dot{\phi}_i + \frac{K_0 f(T)}{T_0}\sum_{r=1}^{M0}\frac{\tau_T^r}{r!}\frac{\partial^r}{\partial t^r}T_{,i}$$

Multiplying the Eq. (114) by $\delta\phi_i$ and integrating over the volume V, we yield

$$\int_V\left(\frac{T_0}{K_0 f(T)}J_i + T_{,i}\right)\delta\phi_i dV = 0 \tag{115}$$

Eq. (115) with the aid of the relation $T_{,i}\delta\phi_i = (T\delta\phi_i)_{,i} - T\delta\phi_{i,i}$ and Gauss's divergence theorem can be written as

$$\int_A T\delta\phi_i n_i dA - \int_V T\delta\phi_{i,i}dV + \frac{T_0}{K_0 f(T)}\int_V J_i\delta\phi_i dV = 0 \tag{116}$$

Using value of $\delta\phi_{i,i}$ in Eq. (116), we obtain

$$\int_A T\delta\phi_i n_i dA + \delta P_1 + \gamma_{10}f\left(T\right)\int_V T\delta e_{kk}dV + d\int_V T\delta P dV - \int_V T\delta Q_1 dV + \delta D_1 = 0 \tag{117}$$

where

$$\delta P_1 = l_1\int_V T\delta T dV, \delta D_1 = \frac{T_0}{K_0 f(T)}\int_V J_i\delta\phi_i dV$$

Introducing a vector function N_i and chemical potential source term M_2 through relations:

$$\eta_i = \dot{N}_i,\ A_0 = \dot{M}_2 \tag{118}$$

Then, the Eqs. (7) and (15) with the aid of (16) and (118) reduce to

$$\delta N_{i,i} = -n\delta P - \gamma_{20}f(T)\delta e_{kk} - d\delta T + \delta M_2 \tag{119}$$

$$\frac{Y_i}{D_0 f(T)} + P_{,i} = 0 \tag{120}$$

where

$$Y_i = \left(\delta^0 + \sum\nolimits_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}\frac{\partial^r}{\partial t^r}\right)\dot{N}_i + D_0 f(T)\sum\nolimits_{r=1}^{M^0}\frac{\tau_P^r}{r!}\frac{\partial^r}{\partial t^r}P_{,i}$$

Multiplying the Eq. (120) by δN_i and integrating over the volume V, we obtain

$$\int_V \left(\frac{Y_i}{D_0 f(T)} + P_{,i} \right) \delta N_i dV = 0 \tag{121}$$

Eq. (121) with the aid of the relation $P_{,i}\delta N_i = (P\delta N_i)_{,i} - P\delta N_{i,i}$ and Gauss's divergence theorem can be written as

$$\int_V \frac{Y_i}{D_0 f(T)} \delta N_i dV + \int_A P\delta N_i n_i dA - \int_V P\delta N_{i,i} dV = 0 \tag{122}$$

Using value of $\delta N_{i,i}$ from Eq. (119) in Eq. (122), we obtain

$$\delta D_2 + \delta P_2 + \int_A P\delta N_i n_i dA + \int_V \gamma_2 P\delta e_{kk} dV + \int_V Pd\delta T dV - \int_V P\delta M_2 dV = 0 \tag{123}$$

where

$$D_2 = \frac{1}{D_0 f(T)} \int_V Y_i \delta N_i dV, \delta P_2 = \int_V nP\delta P dV$$

Eq. (111) with the aid of Eqs. (117) and (123) can be written as

$$\begin{aligned} &\delta \left(\mathrm{W} + D_1 + P_1 + D_2 + P_2 + d \int_V PTdV \right) \\ &= \int_V \rho \left(F_i - \ddot{u}_i + (\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i + (2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})_i \right) \delta u_i dV + \int_A h_i \delta u_i dA - \int_A T\delta \phi_i n_i dA \\ &- \int_A P\delta N_i n_i dA + \int_V T\delta Q_1 dV + \int_V P\delta M_2 dV \end{aligned} \tag{124}$$

The variational principle for a thermoelastic diffusion body with multiple phase delays and temperature-dependent properties in a rotating frame of reference is represented by Eq. (124). Whereas L.H.S. investigates the variation of the sum of the dissipation function, heat potential, chemical potential, and the work of deformation, R.H.S. interprets Eq. (124) as the work performed by the body force, surface traction, inertia forces, heating of the surface, heat source, and mass diffusion source with the virtual deformation.

3 Applications

Imagine an infinite isotropic thermoelastic medium without rotation, where body forces, heat sources, and chemical potential sources are all working solely in a confined region, and surface integrals have been eliminated in (101).

Also considering $C_r = \{F_i^{(r)}(\boldsymbol{x},t), h_i^{(r)}(\boldsymbol{x},t), U_i^{(r)}(\boldsymbol{x},t), \upsilon^{(r)}(\boldsymbol{x},t), Q^{(r)}(\boldsymbol{x},t), \Omega^{(r)}(\boldsymbol{x},t), A_0^{(r)}(\boldsymbol{x},t)\}$ and thus, we get a relation:

$$s\rho T_0 \int_V \left(F_i^{(1)} u_i^{(2)} - F_i^{(2)} u_i^{(1)} \right) dV - \int_V (Q^{(1)} T^{(2)} - Q^{(2)} T^{(1)}) dV - T_0 \int_V (A_0^{(1)} P^{(2)} - A_0^{(2)} P^{(1)}) dV = 0 \tag{125}$$

For the implementation of the inverse Laplace transform defined by (102) in Eq. (125), determine

$$\rho T_0 \int_V \int_0^t F_i^{(1)}(\boldsymbol{x}, t-\zeta) \frac{\partial}{\partial \zeta} u_i^{(2)}(\boldsymbol{x}, \zeta)\, d\zeta\, dV - \int_V \int_0^t Q^{(1)}(\boldsymbol{x}, t-\zeta)\, T^{(2)}(\boldsymbol{x}, \zeta)\, d\zeta\, dV - T_0$$
$$\int_V \int_0^t A_0^{(1)}(\boldsymbol{x}, t-\zeta)\, P^{(2)}(\boldsymbol{x}, \zeta)\, d\zeta\, dV = S_{21}^{12} \tag{126}$$

Here S_{21}^{12} demonstrates the similar expression on the left-hand side barring the superscripts (1) and (2) can be reciprocated [63].

(a) Assuming the point η_1 and η_2, the instantaneous concentrated body forces $F_i^{(1)}$ and $F_i^{(2)}$acting in the direction of $\boldsymbol{x}_j$-axis and $\boldsymbol{x}_k$-axis, respectively. Thus, we obtain

$$C_1 = \left\{F_i^{(1)}, 0, 0, 0, 0, 0, 0\right\},\ C_2 = F_i^{(2)}, 0, 0, 0, 0, 0, 0\},\ F_i^{(1)} = \delta(\boldsymbol{x} - \eta_1)\delta_{ij}\delta(\mathrm{t}),\ F_i^{(2)} = \delta(\boldsymbol{x} - \eta_2)\delta_{ij}\delta(\mathrm{t}),$$
$$u_i^{(1)} \equiv u_i^{(1)}(\boldsymbol{x}, \eta_1, t),\ u_i^{(2)} \equiv u_i^{(2)}(\boldsymbol{x}, \eta_2, t) \tag{127}$$

Incorporating Eq. (127) in Eq. (126), yield

$$\dot{u}_k^{(1)}(\eta_2, \eta_1, t) = \dot{u}_j^{(2)}(\eta_1, \eta_2, t) \tag{128}$$

The displacement $u_j^{(2)}$ at η_1 caused by instantaneous concentrated forces acting at η_2 in the direction of the $\boldsymbol{x}_k$-axis means that the velocity of the displacement $u_k^{(1)}$ at η_2 caused by concentrated forces acting at η_1 in the direction of the $\boldsymbol{x}_j$-axis is equal to that displacement.

(b) Presuming the point η_1 and η_2, there act instantaneous concentrated heat sources $Q^{(1)}$ and $Q^{(2)}$. Thus we obtain

$$\mathrm{C}_1 = \{0, 0, 0, 0, \mathrm{Q}^{(1)}, 0, 0\},\ \mathrm{C}_2 = \{0, 0, 0, 0, \mathrm{Q}^{(2)}, 0, 0\},\ \mathrm{Q}^{(1)} = \delta(\mathrm{x} - \eta_1)\delta(\mathrm{t}),\ \mathrm{Q}^{(2)} = \delta(\mathrm{x} - \eta_2)\delta(\mathrm{t}),$$
$$\mathrm{T}^{(1)} \equiv \mathrm{T}^{(1)}(\mathrm{x}, \eta_1, \mathrm{t}),\ \mathrm{T}^{(2)} \equiv \mathrm{T}^{(2)}(\mathrm{x}, \eta_2, \mathrm{t}) \tag{129}$$

Incorporating Eq. (129) in Eq. (126), yield

$$T^{(1)}(\eta_2, \eta_1, t) = T^{(2)}(\eta_1, \eta_2, t) \tag{130}$$

Finally, it may be said that the temperature $T^{(2)}$ at η_1 caused by heat sources acting at η_2 is equal to the temperature $T^{(1)}$ at η_2 caused by heat sources acting at η_1.

(c) Considering the point η_1 and η_2, there act instantaneous concentrated chemical potential sources $A_0^{(1)}$ and $A_0^{(2)}$. Thus, we obtain

$$C_1 = \left\{0, 0, 0, 0, 0, 0, A_0^{(1)}\right\},\ C_2 = \left\{0, 0, 0, 0, 0, 0, A_0^{(2)}\right\},\ A_0^{(1)} = \delta(\boldsymbol{x} - \eta_1)\,\delta(\mathrm{t}),\ A_0^{(2)} = \delta(\boldsymbol{x} - \eta_2)\,\delta(\mathrm{t}),$$
$$P^{(1)} \equiv P^{(1)}(\boldsymbol{x}, \eta_1, t),\ P^{(2)} \equiv P^{(2)}(\boldsymbol{x}, , \eta_2, t) \tag{131}$$

Incorporating Eq. (131) in Eq. (126), the yield

$$P^{(1)}(\eta_2, \eta_1, t) = P^{(2)}(\eta_1, \eta_2, t) \tag{132}$$

Thus, the chemical potential $P^{(2)}$ created at point η_1 by chemical potential sources acting at a point η_2 due to their action is identical to the chemical potential $P^{(1)}$ produced at point η_2 by chemical potential sources acting at the point η_1.

(d) Taking the point η_1 and η_2, there are instantaneous concentrated body forces $F_i^{(1)}$ acts in the direction $\boldsymbol{x}_j$-axis and heat sources $Q^{(2)}$, respectively. Thus, we obtain

$$C_1 = \left\{F_i^{(1)}, 0, 0, 0, 0, 0, 0\right\},\ C_2 = \left\{0, 0, 0, 0, Q^{(2)}, 0, 0\right\},\ F_i^{(1)} = \delta\left(\boldsymbol{x} - \eta_1\right)\delta_{ij}\delta\left(\mathrm{t}\right),\ Q^{(2)} = \delta\left(\boldsymbol{x} - \eta_2\right)\delta\left(\mathrm{t}\right),$$
$$F_i^{(2)} = 0,\ Q^{(1)} = 0,\ u_i^{(2)} \equiv u_i^{(2)}\left(\boldsymbol{x}, \eta_2, t\right),\ T^{(1)} \equiv T^{(1)}(\boldsymbol{x}, \eta_1, t) \tag{133}$$

Incorporating Eq. (133) in Eq. (126), yield

$$T^{(1)}(\eta_2, \eta_1, t) = -\rho T_0 \dot{u}_j^{(2)}\left(\eta_1, \eta_2, t\right) \tag{134}$$

where $T^{(1)}(\eta_2, \eta_1, t)$ is the temperature at η_2 due to $F_i^{(1)}$ acting at η_1 in the direction $\boldsymbol{x}_j$-axis and $\dot{u}_j^{(2)}(\eta_1, \eta_2, t)$ is the velocity of the particle at η_1 due to $Q^{(2)}$ located at η_2.

Imaging the point η_1 and η_2, there act instantaneous concentrated body forces $F_i^{(1)}$ acts in the direction $\boldsymbol{x}_j$-axis and chemical potential sources $A_0^{(2)}$, respectively. Thus we obtain

$$C_1 = \{F_i^{(1)}, 0, 0, 0, 0, 0, 0\},\ C_2 = \{0, 0, 0, 0, 0, 0, A_0^{(2)}\},\ F_i^{(1)} = \delta(x - \eta_1)\delta_{ij}\delta(t),$$
$$A_0^{(2)} = \delta(x - \eta_2)\delta(t),\ F_i^{(2)} = 0, A_0^{(1)} = 0, u_i^{(2)} \equiv u_i^{(2)}(x, \eta_2, t),\ P^{(1)} \equiv P^{(1)}(x, \eta_1, t) \tag{135}$$

Incorporating Eq. (135) in Eq. (126), yield

$$P^{(1)}(\eta_2, \eta_1, t) = -\rho\dot{u}_j^{(2)}(\eta_1, \eta_2, t) \tag{136}$$

where $P^{(1)}(\eta_2, \eta_1, t)$ is the chemical potential at η_2 due to $F_i^{(1)}$ acting at η_1 in the direction$\boldsymbol{x}_j$-axis and $\dot{u}_j^{(2)}(\eta_1, \eta_2, t)$ is the velocity of particle at η_1 due to $A_0^{(2)}$ located at η_2.

(e) Considering the point η_1 and η_2, there act instantaneous concentrated heat sources $Q^{(1)}$ and instantaneous chemical potential $A_0^{(2)}$. Thus, we obtain

$$C_1 = \{0, 0, 0, 0, Q^{(1)}, 0, 0\},\ C_2 = \{0, 0, 0, 0, 0, 0, A_0^{(2)}\},\ Q^{(1)} = \delta(\boldsymbol{x} - \eta_1)\delta(\mathrm{t}),\ A_0^{(2)} = \delta(\boldsymbol{x} - \eta_2)\delta(\mathrm{t}),$$
$$P^{(1)} \equiv P^{(1)}(\boldsymbol{x}, \eta_1, t),\ T^{(2)} \equiv T^{(2)}(\boldsymbol{x}, \eta_2, t),\ Q^{(2)} = A_0^{(1)} = 0 \tag{137}$$

Incorporating (137) in Eq. (126), yield

$$T_0 P^{(1)}(\eta_2, \eta_1, t) = T^{(2)}(\eta_1, \eta_2, t) \tag{138}$$

The temperature $T^{(2)}$ at point η_1 caused by chemical sources acting at a point η_2 is found to be equivalent to the chemical potential $P^{(1)}$ at point η_2 created by heat sources acting at a point η_1.

(f) Consider heat source $Q^{(1)}$ moving with uniform speed v in an infinite thermoviscoelastic medium in the direction $\boldsymbol{x}_3$ and an instantaneous concentrated heat source $Q^{(2)}$ located at η_2, thus we have

$$C_1 = \left\{0, 0, 0, 0, Q^{(1)}, 0, 0\right\},\ C_2 = \left\{0, 0, 0, 0, Q^{(2)}, 0, 0\right\},\ Q^{(1)} = \delta\left(x_1\right)\delta\left(x_2\right)\delta\left(x_3 - \mathrm{vt}\right),$$
$$Q^{(2)} = \delta\left(\boldsymbol{x} - \eta_2\right)\delta\left(\mathrm{t}\right),\ T^{(1)} \equiv T^{(1)}\left(\boldsymbol{x}, t\right), T^{(2)} \equiv T^{(2)}(\boldsymbol{x}, \eta_2, t) \tag{139}$$

Incorporating Eq. (139) in Eq. (126), yield

$$T^{(1)}(\eta_2, t) = \int_0^t T^{(2)}(\boldsymbol{\zeta}, \eta_2, t - \tau)d\tau \tag{140}$$

where $\boldsymbol{\zeta} = (0, 0, vt)$

In terms of the temperature $T^{(2)}$ owing to the instantaneous concentrated heat source $Q^{(2)}$, there is also the temperature field $T^{(1)}$ caused by the moving heat source $Q^{(1)}$.

(g) To evaluate the velocity field due to moving heat source $Q^{(1)}$ and the instantaneous concentrated body force $F_i^{(2)}$acting at point η_2 in the direction of $\boldsymbol{x}_j$-axis, consider

$$C_1 = \{0, 0, 0, 0, Q^{(1)}, 0, 0\},\ C_2 = \{F_i^{(2)}, 0, 0, 0, 0, 0, 0\},\ Q^{(1)} = \delta(x_1)\delta(x_2)\delta(x_3 - \mathrm{vt}),$$
$$F_i^{(2)} = \delta(\boldsymbol{x} - \eta_2)\delta_{ij}\delta(\mathrm{t}),\ Q^{(2)} = F_i^{(1)} = 0, u_i^{(1)} \equiv u_i^{(1)}(\boldsymbol{x}, t), T^{(2)} \equiv T^{(2)}(\boldsymbol{x}, \eta_2, t) \tag{141}$$

Incorporating Eq. (141) in Eq. (126), yield

$$\dot{u}_j^{(1)}(\eta_2, t) = -\frac{1}{\rho T_0}\int_0^t T^{(j)}\,(\zeta, \eta_2, t - \tau)\,d\tau \tag{142}$$

where the instantaneous concentrated body force $F_i^{(2)}$ operating at point η_2 in the direction of the x_j-axis causes the temperature at point $\mathbf{x}$ to be $T^{(j)}(\boldsymbol{x}, \eta_2, t)$.

(h) Consider chemical potential source $A_0^{(1)}$ moving with uniform speed v in an infinite thermoviscoelastic medium in the direction $\boldsymbol{x}_3$ and an instantaneous concentrated chemical potential source $A_0^{(2)}$ located at η_2, thus we have

$$C_1 = \{0, 0, 0, 0, 0, 0, A_0^{(1)}\},\ C_2 = \{0, 0, 0, 0, 0, 0, A_0^{(2)}\},\ A_0^{(1)} = \delta(x_1)\delta(x_2)\delta(x_3 - \mathrm{vt}),\ A_0^{(2)} = \delta(\boldsymbol{x} - \eta_2)\delta(\mathrm{t}),$$
$$P^{(1)} \equiv P^{(1)}(\boldsymbol{x}, t),\ P^{(2)} \equiv P^{(2)}(\boldsymbol{x}, \eta_2, t) \tag{143}$$

Incorporating Eq. (143) in Eq. (126), yield

$$P^{(1)}\,(\eta_2, t) = \int_0^t P^{(2)}(\boldsymbol{\zeta}, \eta_2, t - \tau)d\tau \tag{144}$$

where

$$\boldsymbol{\zeta} = (0, 0, vt)$$

In terms of the chemical potential $P^{(2)}$ owing to the instantaneous concentrated chemical potential source $A_0^{(2)}$, there is also the chemical potential field $P^{(1)}$ due to the moving chemical potential source $A_0^{(1)}$.

(i) To evaluate the velocity field due to moving chemical potential source $A_0^{(1)}$ and the instantaneous concentrated body force $F_i^{(2)}$acting at point η_2 in the direction of $\boldsymbol{x}_j$-axis, consider

$$C_1 0, = \left\{0, 0, 0, 0, 0, A_0^{(1)}\right\},\ C_2 = \left\{F_i^{(2)}, 0, 0, 0, 0, 0, 0\right\},\ A_0^{(1)} = \delta(x_1)\delta(x_2)\delta(x_3 - \mathrm{vt})$$
$$F_i^{(2)} = \delta(\boldsymbol{x} - \eta_2)\delta_{ij}\delta(\mathrm{t}),\ A_0^{(2)} = F_i^{(1)} = 0,\ u_i^{(1)} \equiv u_i^{(1)}(\boldsymbol{x}, t)$$
$$P^{(2)} \equiv P^{(2)}(\boldsymbol{x}, \eta_2, t) \tag{145}$$

Incorporating Eq. (145) in Eq. (126), yield

$$\dot{u}_j^{(1)}(\eta_2, t) = -\frac{1}{\rho}\int_0^t P^{(j)}\,(\zeta, \eta_2, t - \tau)\,d\tau \tag{146}$$

where the chemical potential at point $\mathbf{x}$ caused by the instantaneous concentrated body force $F_i^{(2)}$ operating at point η_2 in the direction of the x_j-axis is represented by $P^{(j)}(\boldsymbol{x}, \eta_2, t)$.

Plane-Wave

In the case of the two-dimensional example, we examine plane wave propagation within the assumed TDMR model without the presence of surface force, body force, thermal source, or chemical potential source. In the $x_1 - x_3$ plane, we consider the displacement vector, temperature field, and chemical potential field as follows for plane wave vibration:

$$\mathbf{u} = (u_1(x_1, x_3, t), 0, u_3(x_1, x_3, t)), \mathrm{T}(x_1, x_3, t), \mathrm{P}(x_1, x_3, t), \Omega = (0, \Omega_2, 0) \tag{147}$$

Recast the following equations using (147) in (17)–(19) in the absence of body force, heat source, and chemical potential source.

$$(\lambda_{o1} + \mu_0) f(\mathrm{T}) \frac{\partial e}{\partial x_1} + \mu_0 f(\mathrm{T}) \delta u_1 - \gamma_{10} f\,(T)\, \frac{\partial \mathrm{T}}{\partial x_1} - \gamma_{20} f\,(T)\, \frac{\partial \mathrm{P}}{\partial x_1} + \rho(\Omega_2^2 u_1 - 2\Omega_2 \dot{u}_3) = \rho \frac{\partial^2 u_1}{\partial t^2} \tag{148}$$

$$(\lambda_o + \mu_0) f(T) \frac{\partial e}{\partial x_3} + \mu_0 f(\mathrm{T}) \delta u_3 - \gamma_{10} f\,(T)\, \frac{\partial \mathrm{T}}{\partial x_3} - \gamma_{20} f(T) \frac{\partial \mathrm{P}}{\partial x_3} + \rho(\Omega_2^2 u_3 + 2\Omega_2 \dot{u}_1) = \rho \frac{\partial^2 u_3}{\partial t^2} \tag{149}$$

$$\left(\delta^0 + \sum\nolimits_{r=1}^{N0} \frac{\tau_q^r}{r!} \frac{\partial^r}{\partial t^r}\right) (l_1 T_o \dot{T} + \gamma_{10} f(T) T_o \dot{e} + T_o d \dot{P}) = K_0 f(T) \left(1 + \sum\nolimits_{r=1}^{M0} \frac{\tau_T^r}{r!} \frac{\partial^r}{\partial t^r}\right) \delta \mathrm{T} \tag{150}$$

$$\left(\delta^0 + \sum\nolimits_{r=1}^{N0} \frac{\tau_\eta^r}{r!} \frac{\partial^r}{\partial t^r}\right) (n \dot{P} + \gamma_{20} f\,(T)\, \dot{e} + d \dot{T}) = D_0 f(T) \left(1 + \sum\nolimits_{r=1}^{M0} \frac{\tau_P^r}{r!} \frac{\partial^r}{\partial t^r}\right) \Delta \mathrm{P} \tag{151}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}, \quad \mathrm{e} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}$$

The following dimensionless quantities are taken:

$$x_i' = \frac{\Omega *}{c_1} x_i, u_i' = \frac{\Omega *}{c_1} u_i, \mathrm{t}' = \Omega * \mathrm{t}, \tau_T' = \Omega * \tau_T, \tau_q' = \Omega * \tau_q,$$

$$\tau_\eta' = \Omega * \tau_\eta, \tau_P' = \Omega * \tau_P, \mathrm{T}' = \frac{\gamma_1}{\rho c_1^2} \mathrm{T}, \mathrm{P}' = \frac{1}{b \gamma_2} \mathrm{P}, \Omega_2' = \frac{\Omega_2}{\Omega *} \tag{152}$$

where

$$\Omega * = \frac{\rho C_E c_1^2}{K}, c_1^2 = \frac{\lambda + 2\mu}{\rho}$$

ω^* is the characteristic frequency and c_1 is the longitudinal wave velocity in the medium.

4 Solution Procedure

The displacement components $u_1\ (x_1, x_3, t)$ and $u_3(x_1, x_3, t)$ have a dimensionless relationship with the scalar potentials $\varphi\ (x_1, x_3, t)$ and $\psi(x_1, x_3, t)$.

$$u_1 = \frac{\partial \varphi}{\partial x_1} \frac{\partial \psi}{\partial x_3}, u_3 = \frac{\partial \varphi}{\partial x_3} + \frac{\partial \psi}{\partial x_1} \tag{153}$$

The motivation for using these potentials is to decouple the governing equations and classify wave types more effectively. The scalar potential φ represents longitudinal motion associated with P-waves, while the vector potential ψ corresponds to shear motion linked to S-waves. This formulation is particularly useful in thermoelastic diffusion, where multiple wave types (thermal, mechanical, and chemical potential waves) exist. By expressing displacement in terms of potentials, we can reduce the complexity of the coupled system and obtain analytical solutions more easily. After suppressing the primes, incorporate the dimensionless quantities from Eq. (152) in Eqs. (148)–(151). These equations can then be decoupled using Eq. (153) and reduced to the form

$$\left((a_{11}+a_{12})\Delta+\Omega_2^2-\frac{\partial^2}{\partial t^2}\right)\varphi-2\Omega_2\frac{\partial\psi}{\partial t}-T-a_{14}\mathrm{P}=0 \tag{154}$$

$$2\Omega_2\frac{\partial\varphi}{\partial t}+\left(a_{12}\delta+\Omega_2^2-\frac{\partial^2}{\partial t^2}\right)\psi=0 \tag{155}$$

$$J_{11}^{*}a_{22}\delta\dot{\varphi}+\left(J_{11}^{*}a_{21}\frac{\partial}{\partial t}-J_{12}^{*}\delta\right)T+J_{11}^{*}a_{23}\dot{P}=0 \tag{156}$$

$$J_{21}^{*}a_{32}\Delta\dot{\varphi}+J_{21}^{*}a_{33}\dot{T}+\left(J_{21}^{*}a_{31}\frac{\partial}{\partial t}-J_{22}^{*}\Delta\right)P=0 \tag{157}$$

where

$$a_{11}=\frac{(\lambda_{o1}+\mu_0)f(\mathrm{T})}{\rho c_1^2},a_{12}=\frac{\mu_0 f(\mathrm{T})}{\rho c_1^2},a_{13}=\frac{b\gamma_{20}^2(f\,(T))^2}{\rho c_1^2},a_{21}=\frac{l_1c_1^2}{K_0f(T)\Omega*}T_o,$$

$$a_{22}=\frac{\gamma_{10}^2f(T)}{K_0\rho\Omega*}T_o,a_{23}=\frac{b\gamma_{10}\gamma_{20}f(T)}{K_0\rho\Omega*}T_od,a_{31}=\frac{nc_1^2}{D_0f(T)\Omega*},$$

$$a_{32}=\frac{c_1^2}{D_0f(T)b\Omega*},a_{33}=\frac{c_1^4d\rho}{D_0b\Omega*\gamma_{10}\gamma_{20}(f\,(T))^3},$$

$$J_{11}^{*}=\left(\delta^0+\sum\nolimits_{r=1}^{N0}\frac{\tau_q^r}{r!}\frac{\partial^r}{\partial t^r}\right),J_{12}^{*}=\left(1+\sum\nolimits_{r=1}^{M0}\frac{\tau_T^r}{r!}\frac{\partial^r}{\partial t^r}\right),$$

$$J_{21}^{*}=\left(\delta^0+\sum\nolimits_{r=1}^{N0}\frac{\tau_\eta^r}{r!}\frac{\partial^r}{\partial t^r}\right),J_{22}^{*}=\left(1+\sum\nolimits_{r=1}^{M0}\frac{\tau_P^r}{r!}\frac{\partial^r}{\partial t^r}\right) \tag{158}$$

We assume the solution of Eqs. (154)–(157) as

$$(\varphi,\psi,T,P)=(\hat{\varphi},\hat{\psi},\widehat{T},\hat{P})e^{-i\xi[(l_1x_1+l_3x_3)+ct]} \tag{159}$$

where $\Omega=\xi c$, ξ is the wave number & c is the phase velocity

The system of four homogeneous equations in four unknowns $\hat{\varphi}$, $\widehat{T}$,$\hat{P}$ and $\hat{\psi}$ are produced by including Eq. (159) into Eqs. (154)–(157). The non-trivial solution of the resulting equations yields the following polynomial equations in ξ as

$$(b_{11}\xi^8+b_{12}\xi^6+b_{13}\xi^4+b_{14}\xi^2+b_{15})=0 \tag{160}$$

where

$$\mathrm{b}_{11} = \mathrm{a}^0_{11}\mathrm{a}_{12}\mathrm{a}^0_{23}\mathrm{a}^0_{34},$$

$$\mathrm{b}_{12} = -\mathrm{a}_{12}\mathrm{a}^0_{12}\mathrm{a}^0_{23}\mathrm{a}^0_{34} - \mathrm{a}^0_{11}\left(-\mathrm{a}_{12}\left(\mathrm{a}^0_{23}\mathrm{a}^0_{33} + \mathrm{a}^0_{22}\mathrm{a}^0_{34}\right) + \mathrm{a}^0_{12}\mathrm{a}^0_{23}\mathrm{a}^0_{34}\right) - \mathrm{a}_{12}\mathrm{a}^0_{21}\mathrm{a}^0_{34} - \mathrm{a}_{13}\mathrm{a}_{12}\mathrm{a}^0_{31}\mathrm{a}^0_{23},$$

$$\mathrm{b}_{13} = [-\mathrm{a}^0_{11}\left(\mathrm{a}^0_{12}\left(\mathrm{a}^0_{23}\mathrm{a}^0_{33} + \mathrm{a}^0_{22}\mathrm{a}^0_{34}\right) - \mathrm{a}_{12}\left(\mathrm{a}^0_{22}\mathrm{a}^0_{33} - \mathrm{a}^0_{32}\mathrm{a}^0_{24}\right)\right) + \mathrm{a}^0_{12}\left(-\mathrm{a}_{12}\left(\mathrm{a}^0_{23}\mathrm{a}^0_{33} + \mathrm{a}^0_{22}\mathrm{a}^0_{34}\right) + \mathrm{a}^0_{12}\mathrm{a}^0_{23}\mathrm{a}^0_{34}\right) -$$
$$4\Omega_2^2\Omega^2\mathrm{a}^0_{23}\mathrm{a}^0_{34} + (\mathrm{a}^0_{12}\mathrm{a}^0_{21}\mathrm{a}^0_{34} - \mathrm{a}_{12}\left(\mathrm{a}^0_{21}\mathrm{a}^0_{33} - \mathrm{a}^0_{31}\mathrm{a}^0_{24}\right) + \mathrm{a}_{13}(\mathrm{a}^0_{12}\mathrm{a}^0_{31}\mathrm{a}^0_{23} - \mathrm{a}_{12}\left(\mathrm{a}^0_{31}\mathrm{a}^0_{22} - \mathrm{a}^0_{21}\mathrm{a}^0_{32}\right)),$$

$$\mathrm{b}_{14} = [-\mathrm{a}^0_{11}(\mathrm{a}^0_{12}\left(\mathrm{a}^0_{22}\mathrm{a}^0_{33} - \mathrm{a}^0_{32}\mathrm{a}^0_{24}\right) + \left(\mathrm{a}^0_{12}\right)^2\left(\mathrm{a}^0_{23}\mathrm{a}^0_{33} + \mathrm{a}^0_{22}\mathrm{a}^0_{34}\right) - \mathrm{a}^0_{12}\mathrm{a}_{12}\left(\mathrm{a}^0_{22}\mathrm{a}^0_{33} - \mathrm{a}^0_{32}\mathrm{a}^0_{24}\right)$$
$$-\ 4\Omega_2^2\Omega^2\left(\mathrm{a}^0_{23}\mathrm{a}^0_{33} + \mathrm{a}^0_{22}\mathrm{a}^0_{34}\right) + \mathrm{a}^0_{12}\left(\mathrm{a}^0_{21}\mathrm{a}^0_{33} - \mathrm{a}^0_{31}\mathrm{a}^0_{24}\right) + \mathrm{a}_{13}\mathrm{a}^0_{12}\left(\mathrm{a}^0_{31}\mathrm{a}^0_{22} - \mathrm{a}^0_{21}\mathrm{a}^0_{32}\right)],$$

$$\mathrm{b}_{15} = (\left(\mathrm{a}^0_{12}\right)^2 - 4\Omega_2^2\Omega^2)\left(\mathrm{a}^0_{22}\mathrm{a}^0_{33} - \mathrm{a}^0_{32}\mathrm{a}^0_{24}\right)$$

and

$$a^0_{11} = a_{11} + a_{12},\ a^0_{12} = \Omega^2 + \Omega_2^2,\ a^0_{21} = i\Omega a_{22}E^*_{11},\ a^0_{22} = -i\Omega a_{21}E^*_{11},\ a^0_{23} = E^*_{12},\ a^0_{24} = -i\Omega a_{23}E^*_{11},\ a^0_{31} = i\Omega$$
$$a_{32}E^*_{21},\ a^0_{32} = -i\Omega a_{33}E^*_{21},\ a^0_{33} = -i\Omega a_{31}E^*_{21},\ a^0_{34} = E^*_{22}$$

$$E^*_{11} = \left(\delta^0 + \sum_{r=1}^{N^0}\frac{(-i\Omega\tau_q)^r}{r!}\right), E^*_{12} = \left(1 + \sum_{r=1}^{M^0}\frac{(-i\Omega\tau_T)^r}{r!}\right),$$

$$E^*_{21} = \left(\delta^0 + \sum_{r=1}^{N^0}\frac{(-i\Omega\tau_\eta)^r}{r!}\right), E^*_{22} = \left(1 + \sum_{r=1}^{M^0}\frac{(-i\Omega\tau_P)^r}{r!}\right) \tag{161}$$

Solving Eq. (160), we obtain eight roots, that is ξ_1, ξ_2, ξ_3 & ξ_4 correspond to a positive x_3 direction and the other four roots $-\xi_1$, $-\xi_2$, $-\xi_3$ & $-\xi_4$correspond to negative x_3 direction. Corresponding to roots ξ_1, ξ_2, ξ_3 & ξ_4, there exist four longitudinal waves in descending order of their velocities namely primary (P) wave, secondary (S) wave, thermal (T) wave, and chemical potential (CP) wave.

For the derivation of phase velocity, attenuation coefficient, specific loss, and penetration depth for wave produced by solving a system of Eqs. (148)–(151) are given as:

(i) Phase velocity

The phase velocity is given by

$$V_j = \frac{\Omega}{\left|Re(\xi_j)\right|},\ \ j = 1, 2, 3, 4 \tag{162}$$

where V_1, V_2, V_3, V_4 are the phase velocities of P, S T, and CP, waves, respectively.

(ii) Attenuation coefficient

The attenuation coefficient are defined as

$$Q_j = Im\left(\xi_j\right), j = 1, 2, 3, 4 \tag{163}$$

where Q_1, Q_2, Q_3 & Q_4 are the attenuation coefficients of P, S, T, and CP waves, respectively.

Special Cases

1. The analogous conclusions for the thermoelastic diffusion model may be derived by using $\delta^0 = 1$, $\tau_T = \tau_q = \tau_\eta = \tau_P = 0$, $\alpha^0 = 0$ and independent of rotation impact in Eqs. (20), (81), (109), (124) and (160). These results are similar to those obtained by Kumar and Vandana [30] as a particular case.
2. The corresponding formulas for thermoelastic diffusion under the L-S model may be found by taking $\delta^0 = 1$, $\tau_T = 0$, $\tau_P = 0$, $N^0 = 1$, $\alpha^0 = 0$ and independent of rotation impact in Eqs. (20), (81), (109) and (124). In a specific instance, these results are comparable to those obtained by Ezzat and Fayik [43].
3. The ensuing results for thermoelastic diffusion with the single-phase-lag model are found in Eqs. (20), (81), (109), (124) and (160), letting $N^0 = 1$, $M^0 = 1$, $\delta^0 = 1$, $\alpha^0 = 0$ and independent of rotation impact.
4. When the comparable results for thermoelastic diffusion with the dual-phase-lag model are obtained using Eqs. (20), (81), (109), (124) and (160) with $\delta^0 = 1$, $N^0 = 2$, $M^0 = 1$, $\alpha^0 = 0$, and independent of rotation impact, these results are identical to those obtained by Kumar and Vandana [30] as a particular case.
5. The results of Eqs. (20), (81), (109), (124) and (160) for thermoelastic models with dual phases and lags, in the absence of a diffusion parameter and independent of the impact of rotation, are comparable to those found by Kumar and Vandana [31] as a particular case.

Numerical Results and Discussion

The physical constants used in this study are taken from [61], which provides well-established values for materials commonly used in thermoelastic diffusion studies. The selected parameters correspond to copper, a material widely studied for its thermal and mechanical properties in rotating thermoelastic systems. Copper exhibits high thermal conductivity and well-characterized thermoelastic behavior, making it an ideal candidate for validating our mathematical model. For numerical calculations, consider the copper material (thermoelastic diffusion solid), which has wide applications in industry. The values of physical constants of the problem were taken as (following [61]):

$$\lambda = 7.76 \times 10^{10}\mathrm{Kg}m^{-1}s^{-2},\ \mu = 3.86 \times 10^{10}\mathrm{Kg}m^{-1}s^{-2},\ T_0 = 293\mathrm{K},\ C_E = 0.3831 \times 10^{3}\mathrm{J}Kg^{-1}K^{-1},$$

$$\alpha_t = 0.0178 \times 10^{-3}K^{-1},\ \alpha_c = 0.0198 \times 10^{-2}m^3Kg^{-1},\ \mathrm{a} = 1.2 \times 10^{4}m^2s^{-2}K^{-1},\ \mathrm{b} = 9 \times 10^{5}Kg^{-1}m^5s^{-2},$$

$$\mathrm{D} = 0.85 \times 10^{-8}Kgm^{-3},\ \rho = 8.954 \times 10^{3}Kgm^{-3},\ \mathrm{K} = 0.386 \times 10^{3}\,Wm^{-1}K^{-1},\ \tau_T = 0.1s,\ \tau_q = 0.07s,$$

$$\tau_P = 0.06s,\ \tau_\eta = 0.09s$$

and temperature dependent and rotation parameters is given by $\alpha^0 = 0.0005\ \mathrm{K}^{-1}$, $\Omega_2 = 0.5$

For multi-phase-lags $N^0 = 4$, $M^0 = 4$.

The attenuation coefficient and phase velocity of plane waves have been calculated using the Matlab 24.1.0.2537033 program. Figs. 1–24 display the phase velocity and attenuation coefficient fluctuations with respect to frequency, respectively.

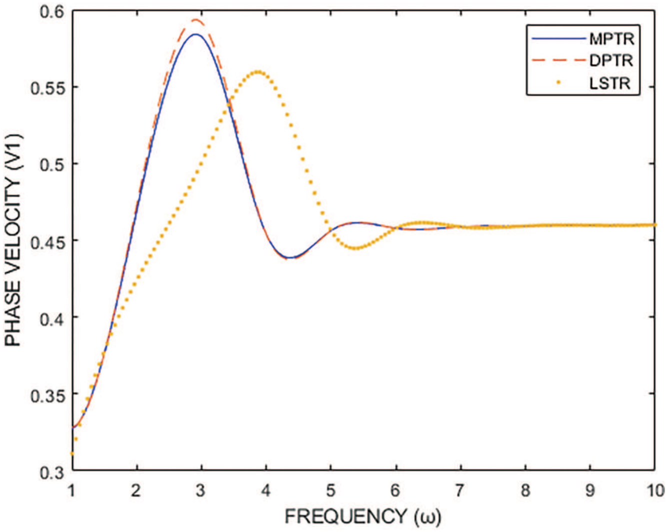

Figure 1: Phase velocity variations V_1 w.r.t. frequency

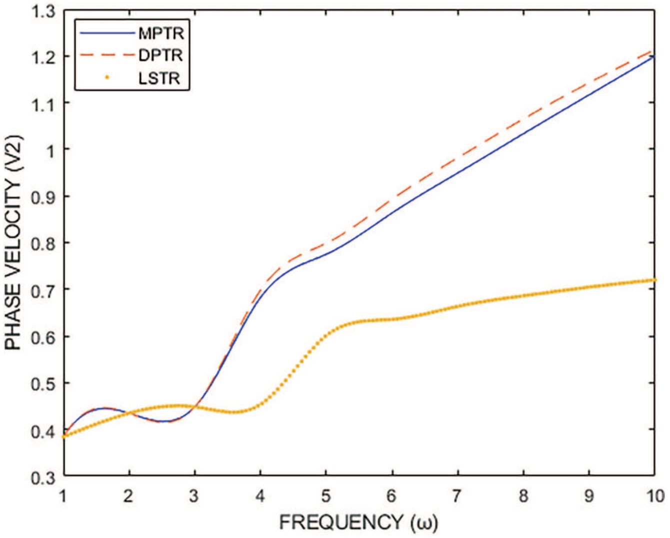

Figure 2: Phase velocity variations V_2 w.r.t. frequency

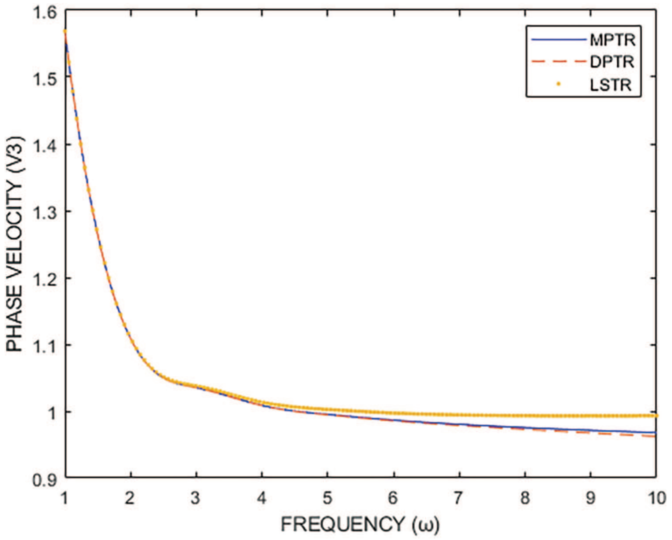

Figure 3: Phase velocity variations V_3 w.r.t frequency

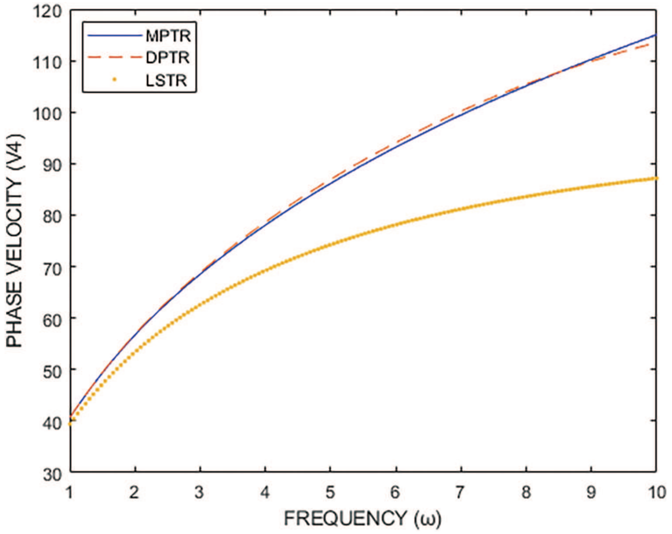

Figure 4: Phase velocity variations V_4 w.r.t. frequency

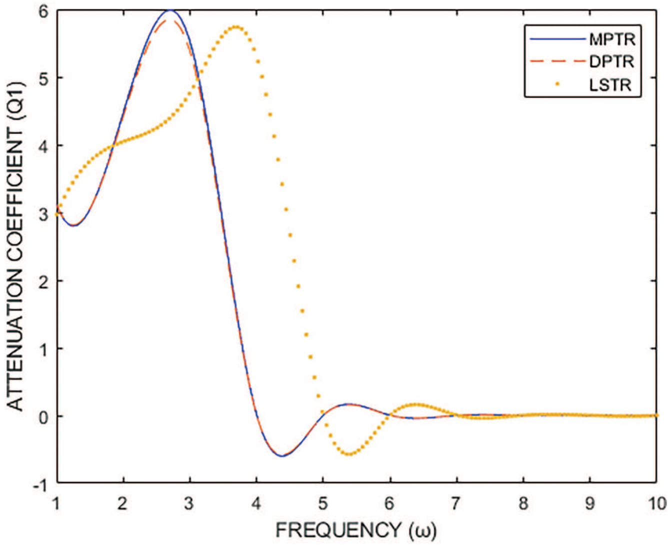

Figure 5: Attenuation coefficient variations coefficient variations

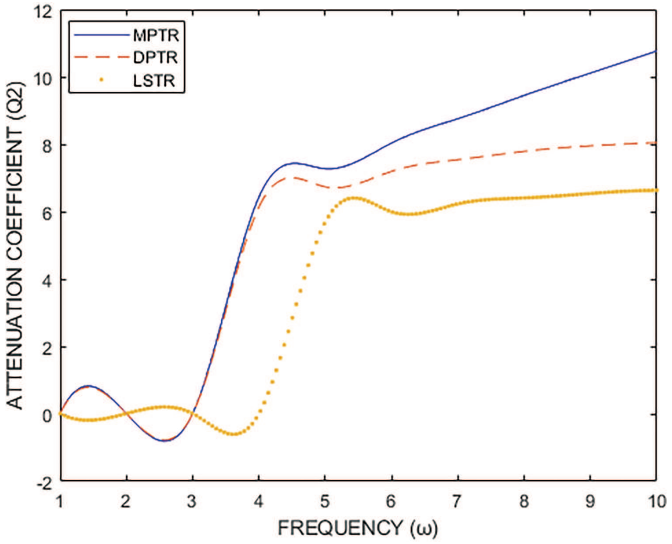

Figure 6: Attenuation Q_1 w.r.t frequency Q_2 w.r.t frequency

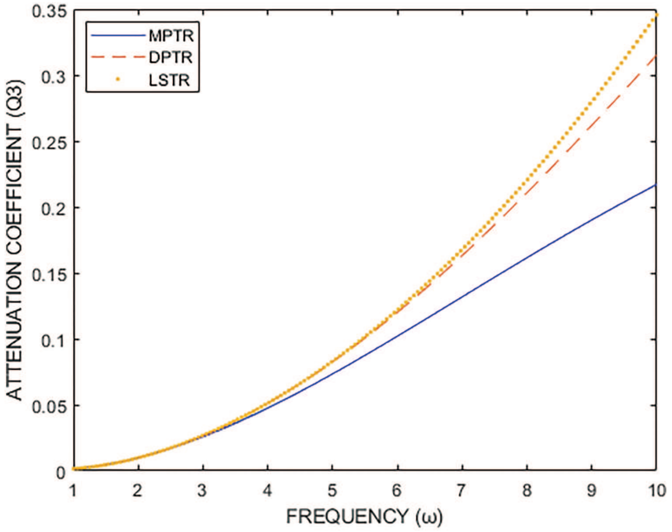

Figure 7: Attenuation coefficient variations Q_3 w.r.t frequency

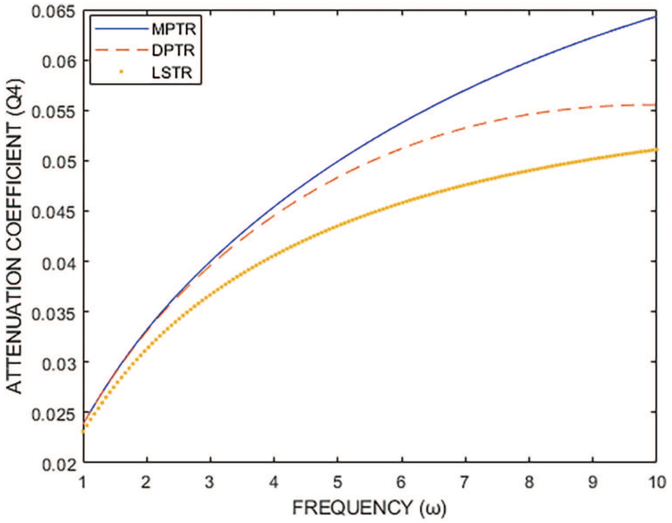

Figure 8: Attenuation coefficient variations Q_4 w.r.t frequency

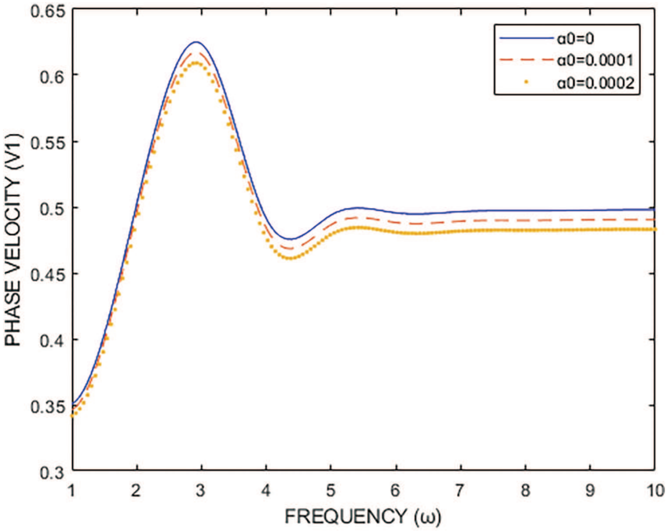

Figure 9: Phase velocity variations V_1 w.r.t frequency

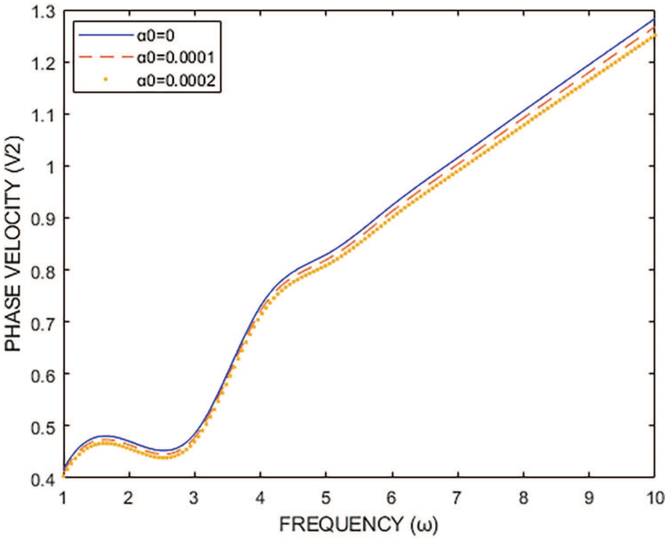

Figure 10: Phase velocity variations V_2 w.r.t frequency

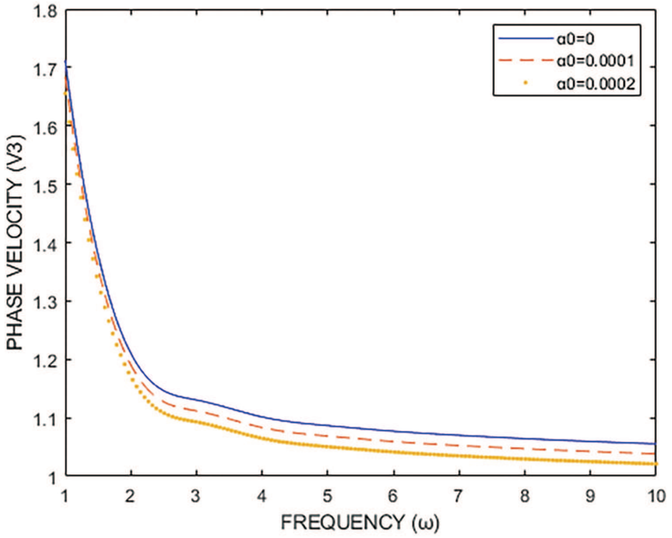

Figure 11: Phase velocity variations V_3 w.r.t frequency

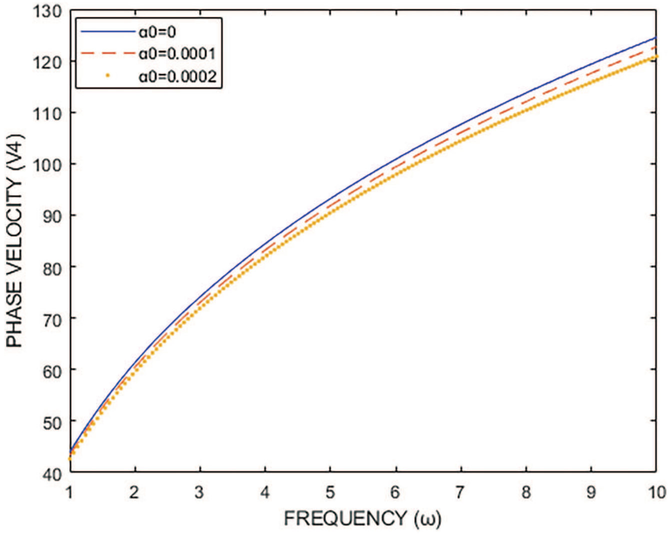

Figure 12: Phase velocity variations V_4 w.r.t frequency

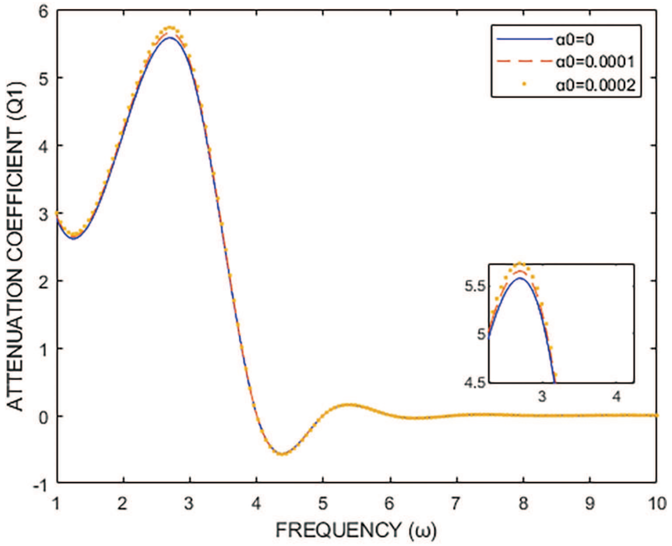

Figure 13: Attenuation coefficient variations Q_1 w.r.t frequency

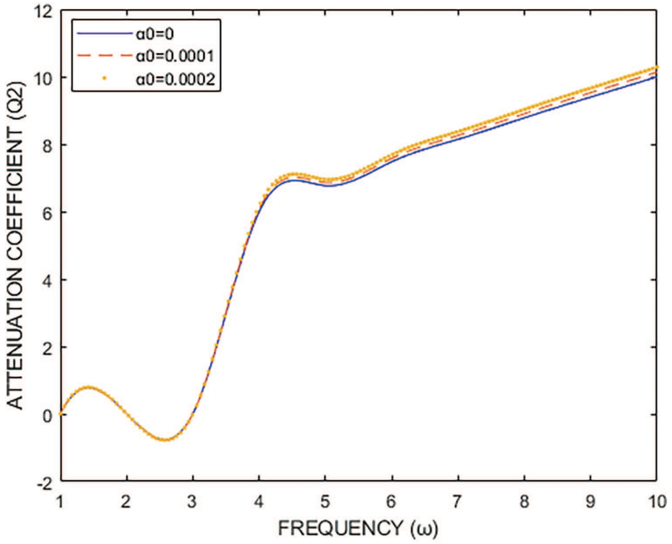

Figure 14: Attenuation coefficient variations Q_2 w.r.t frequency

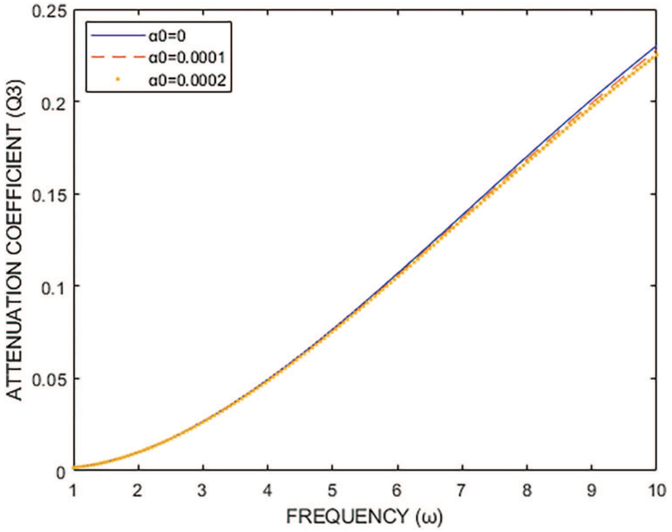

Figure 15: Attenuation coefficient variations Q_3 w.r.t frequency

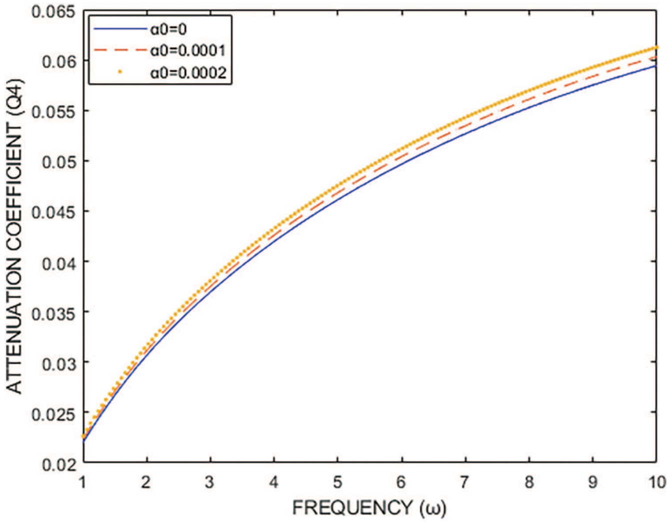

Figure 16: Attenuation coefficient variations Q_4 w.r.t frequency

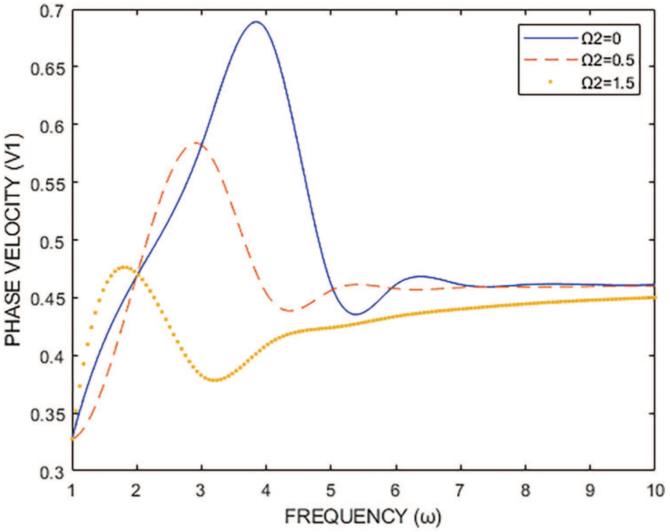

Figure 17: Phase velocity variations V_1 w.r.t frequency

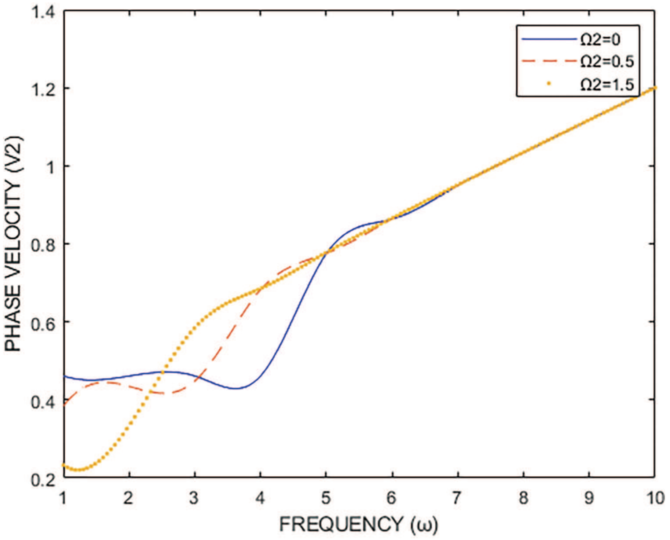

Figure 18: Phase velocity variations V_2 w.r.t frequency

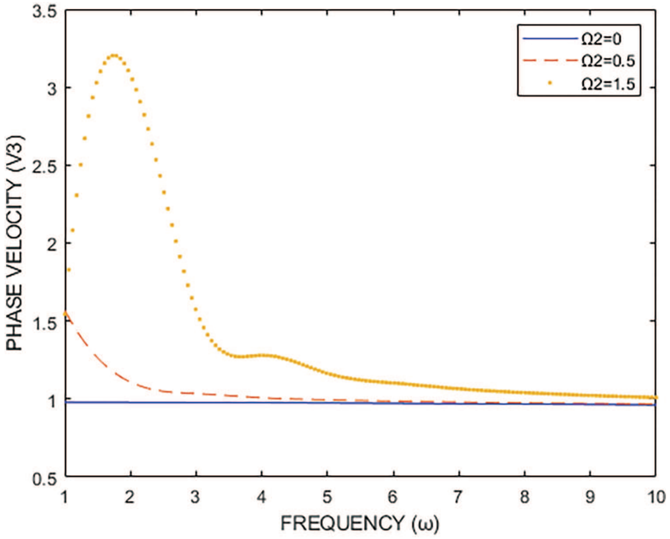

Figure 19: Phase velocity variations V_3 w.r.t frequency

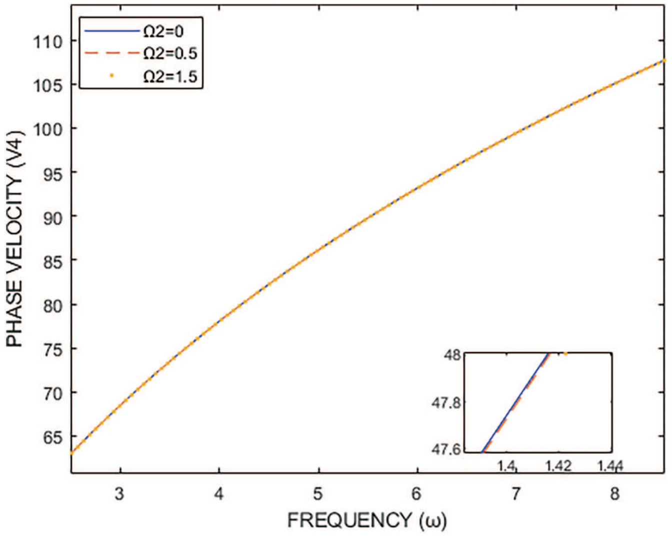

Figure 20: Phase velocity variations V_4 w.r.t frequency

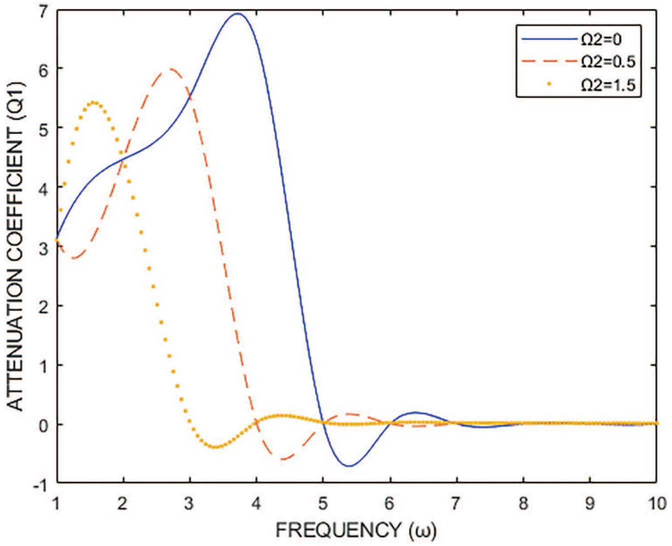

Figure 21: Attenuation coefficient variations Q_1 w.r.t frequency

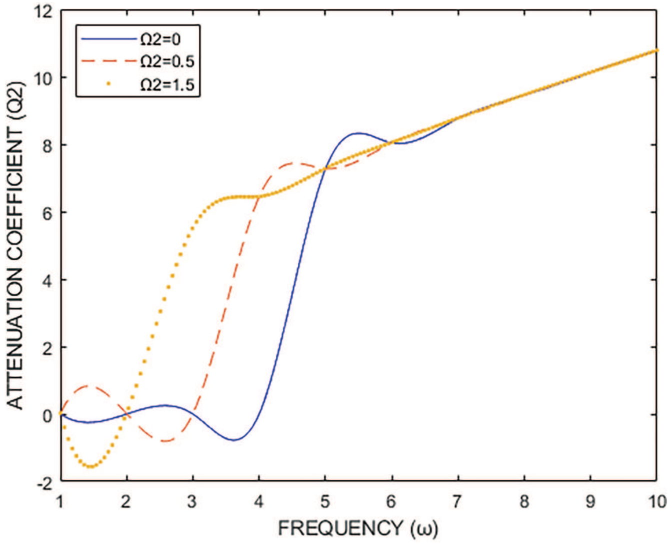

Figure 22: Attenuation coefficient variations Q_2 w.r.t frequency

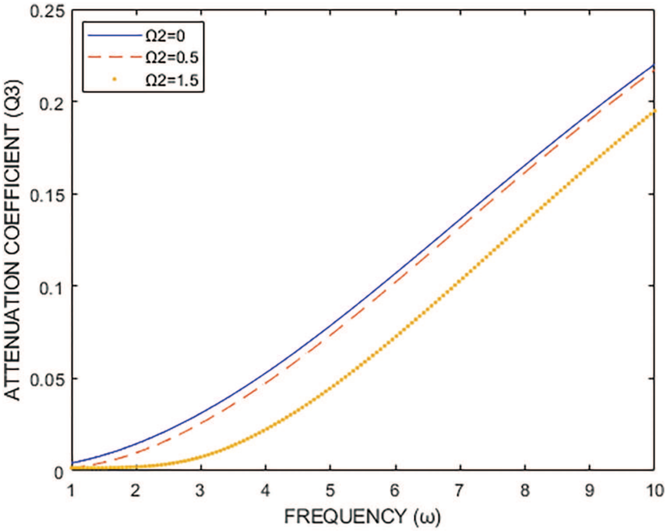

Figure 23: Attenuation coefficient variations Q_3 w.r.t frequency

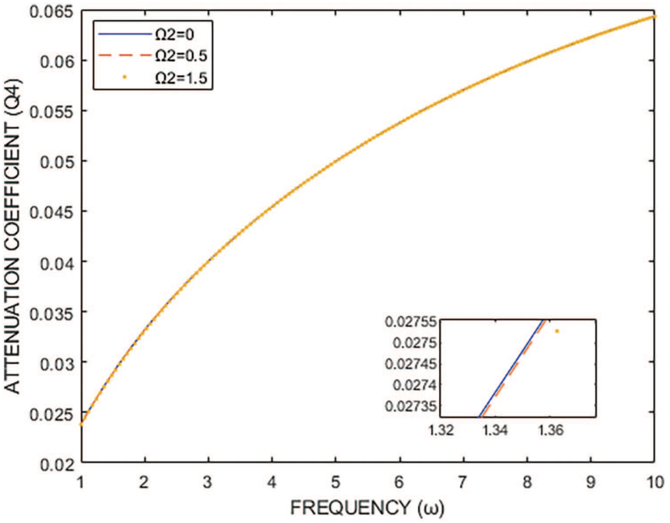

Figure 24: Attenuation coefficient variations Q_4 w.r.t frequency

The multi-phase-lags model under temperature dependent property and rotation (MPTR) is represented by the solid line (__) in Figs. 1–8, the dual-phase-lag model under temperature dependent property and rotation (DPTR) is represented by the small dashed line (–), and the LS model under temperature dependent property and rotation (LSTR) is represented by the dotted line (. . .). The multi-phase-lags model under temperature dependent property and rotation (MPTR) for $\alpha^0 = 0$ is represented by solid line (__) in Figs. 9–16, MPTR for $\alpha^0 = 0.0001$ is represented by small dash line (–), and MPTR for $\alpha^0 = 0.0002$ is represented by dotted line (. . .). The multi-phase-lags model under temperature dependent property and rotation (MPTR) for $\Omega_2 = 0$ is shown by the solid line (__) in Figs. 17–24, the MPTR for $\Omega_2 = 0.5$ is represented by the small dashed line (–), and the MPTR for $\Omega_2 = 1.5$ is represented by the dotted line (. . .).

Phase velocity: Fig. 1 shows that the phase velocity V_1 values for MPTR, DPTR, and LSTR increase with frequency at first for $0 \leq \omega < 3$, $0 \leq \omega < 3$, and $0 \leq \omega < 3.8$, respectively. After that, they exhibit oscillatory behavior and begin to converge to the boundary surface. According to Fig. 2, all models exhibit fluctuating phase velocity V_2 values at first, but when $\omega > 6$, these values increase fast. Figs. 3–4 show the dominant behavior of the LSTR model, where velocity V_4 increases rapidly with frequency rise and velocity V_3 values drop with frequency. For all models $\alpha^0 = 0$, $\alpha^0 = 0.0001$, and $\alpha^0 = 0.0002$, the values of phase velocities V_1 and V_2 exhibit oscillatory behavior with a predominating effect of V_1 and $V_2$for $\alpha^0 = 0$ with an increase in frequency (Figs. 9 and 10). For all models with $\alpha^0 = 0$, $\alpha^0 = 0.0001$, and $\alpha^0 = 0.0002$, Figs. 11 and 12 show that the values of phase velocity V_3 fall as frequency increases while values of V_4 increase. The phase velocity V_1 values for $\Omega_2 = 0$, $\Omega_2 = 0.5$, and $\Omega_2 = 1.5$ are seen in Figs. 17 and 18 to exhibit oscillatory behavior with increasing frequency, while the phase velocity V_2 values initially exhibit oscillatory behavior for $\omega > 7$ values of velocity for all three model overlaps. Figs. 19 and 20 illustrates how phase velocity V_3 values for $\Omega_2 = 0$ and $\Omega_2 = 0.5$ fall as frequency increases, while they first increase for $0 \leq \omega < 2$ and then gradually decline for $\Omega_2 = 1.5$. As seen in Fig. 20, the values of V_4 grow quickly with an increase in frequency for $\Omega_2 = 0$, $\Omega_2 = 0.5$, and $\Omega_2 = 1.5$.

Attenuation coefficient: As the frequency increases, the attenuation coefficients Q_1 and Q_2 exhibit oscillatory behavior, whereas Q_3 and Q_4 show a smooth increase with frequency (Figs. 5–8) for all models (MPTR, DPTR, LSTR). In Figs. 13–16, the attenuation coefficients $Q_1$1 and Q_2 exhibit fluctuating behavior, whereas Q_3 and Q_4 increase rapidly with an increase in frequency for all models with $\alpha^0 = 0$, $\alpha^0 = 0.0001$, and $\alpha^0 = 0.0002$. Figs. 21–24 illustrate that attenuation coefficients Q_1 and Q_2 exhibit oscillatory behavior at first for $\Omega_2 = 0$, $\Omega_2 = 0.5$, and $\Omega_2 = 1.5$, and that they display the same values for all three models for $\omega > 7$. In contrast, attenuation coefficients Q_3 and Q_4 gradually increase with increasing frequency.

Our numerical example, while mathematically detailed, serves to highlight several key physical phenomena that standard Fourier-based thermoelastic models cannot capture. For instance, by incorporating multi-phase-lag effects, our model predicts finite thermal propagation speeds, which in turn influence the dispersion and attenuation characteristics of thermoelastic waves in copper. These features suggest that the dynamic response of the material under transient thermal and mechanical loads is more complex than what is indicated by classical theories. Standard models, which assume an instantaneous heat flux response, would not be able to describe these delayed interactions and the resulting non-standard dispersion behavior.

Moreover, although direct experimental evidence for these specific predictions in copper is still limited, analogous behaviors have been observed in other advanced thermoelastic systems and materials where non-Fourier heat conduction effects are present. Model validation can be approached through sensitivity analyses and benchmarking against experimental trends reported in the literature. Such comparisons provide a means for modelers to assess the realism and accuracy of the predictions. In summary, our numerical calculations illustrate that the integration of rotation, multi-phase-lag, and temperature-dependent properties introduces significant modifications in wave propagation that could be critical for designing systems in high-speed or high-temperature environments, and they set the stage for future experimental studies to validate these theoretical predictions.

Table 1 summarizes the modeling approach, inclusion of rotational effects, phase-lag formulations, temperature-dependent properties, derivation of fundamental theorems, wave propagation analysis, and experimental relevance, with corresponding citations to related literature:

Table 1: Comparison of key aspects of the present work with previous thermoelastic diffusion models and theories

Aspect	This work	Previous works
Modeling approach	Integrates a multi-phase-lag thermoelastic diffusion model with temperature-dependent properties in a rotating coordinate system.	Traditional models typically assume constant material properties and a stationary frame (e.g., Lord & Shulman, 1967; Green & Lindsay, 1972).
Inclusion of rotational effects	Incorporates Coriolis and centripetal accelerations, capturing inertial effects on wave propagation.	Most previous studies neglect rotation or assume it is negligible (e.g., Nowacki, 1974; Biot, 1956).
Phase-lag formulation	Employs a refined multi-phase-lag approach (beyond single- and dual-phase-lag models) to account for finite propagation speeds and delayed responses in both heat and mass diffusion.	Earlier works often rely on single-phase-lag (Fourier, 1822; Cattaneo, 1958) or dual-phase-lag formulations (Tzou, 1995; Roychoudhuri, 2007).
Temperature-dependent properties	Material properties (e.g., elastic modulus, thermal conductivity) vary with temperature, enhancing model realism for high-temperature applications.	Previous studies usually assume constant properties, limiting their applicability in non-isothermal environments (Noda, 1991; Ezzat et al., 2004).
Fundamental theorems	Derives and validates energy, uniqueness, reciprocity, and variational principles within the extended model, ensuring mathematical consistency.	Fundamental theorem derivations are available in classical thermoelasticity (Biot, 1956; Nowacki, 1974), but rarely in the context of combined rotation and multi-phase-lag effects.
Wave propagation analysis	Predicts finite propagation speeds, complex dispersion, and enhanced attenuation of thermoelastic waves in copper, emphasizing effects not captured by standard Fourier models.	Conventional models predict instantaneous thermal response and simpler wave behavior without the detailed dispersion and attenuation characteristics (Kumar & Gupta, 2013).

(Continued)

Table 1 (continued)

Aspect	This work	Previous works
Experimental relevance	Discusses potential experimental validations and benchmarking strategies, comparing predicted trends with reported experimental observations in advanced materials.	Limited experimental validation is provided in earlier works, which often focus on mathematical and numerical exercises (Abouelregal & Sedighi, 2021).

5 Conclusion

This work provides a new theoretical and numerical framework for analyzing thermoelastic diffusion with multi-phase delays in a rotating medium. We have derived and validated fundamental theorems for energy, uniqueness, reciprocity, and variational criteria. Additionally, our numerical results reveal the impact of rotation, temperature-dependent properties, and phase-lag effects on wave propagation, showing that:

- The inclusion of multi-phase-lag effects significantly alters wave speeds and attenuation compared to single-phase-lag and dual-phase-lag models.
- The influence of temperature-dependent material properties introduces nonlinear variations in wave behavior that cannot be captured by conventional models.
- Rotation effects modify the dispersion characteristics of waves, which is crucial for applications in rotating semiconductor materials and geophysical modeling.

The uniqueness and reciprocity theorems are demonstrated by the use of the Laplace transform. The energy theorem can be used as a guide to study a system's motion under various sources, including mechanical force, thermal source, and mass-diffusion source. Instantaneous concentrated body forces, heat forces, chemical potential sources, and moving heat and chemical potential sources are treated as applications of the reciprocity theorem. To ascertain the existence and characteristics of mathematical objects, the uniqueness theorem is important. Differential equations of thermoelasticity can be integrated using the Green function, as uniquely inferred from the reciprocity theorem. It is also possible to infer alternative approximation strategies for the solution of various issues (statics and dynamics) thanks to the variational criteria, which aids in the easy aspect of deriving the differential equation of various systems, such as membranes and cells. The two-dimensional case of the postulated model is also analyzed in terms of wave propagation. There are discovered to be four interconnected longitudinal waves: the primary (P) wave, secondary (S) wave, thermal (T) wave, and chemical potential (CP) wave. The numerical computation and graphical presentation of wave properties, such as phase velocity and attenuation coefficient, are described. For all models (MPTR, DPTR, and LSTR), the magnitude of phase velocities V_1 and V_2 exhibit oscillatory behaviors, while the values of phase velocity V_3 indicate opposing behaviors to V_4. Phase velocities V_1, V_2, V_3, and V_4 exhibit dominant behavior when a temperature-dependent parameter is absent. A variable reaction is observed for the phase velocities V_1, V_2, V_3, and V_4 when the rotation parameter is varied. While the amount of attenuation coefficients Q_3 & Q_4 increases with an increase in frequency for MPTR, DPTR, and LSTR, there is a change in the behavior of attenuation coefficients Q_1 & Q_2. The attenuation coefficients Q_1, Q_2, & Q_4

for $\alpha^0 = 0.0002$ maintains a larger magnitude as the frequency increases compared to Q_1, Q_2, & Q_4 for $\alpha^0 = 0$, $\alpha^0 = 0.0001$, and Q_3 maintains a smaller magnitude. When frequency increases, attenuation coefficients Q_1, Q_2, Q_3, & Q_4 exhibit mixed behavior for $\Omega_2 = 0$, $\Omega_2 = 0.5$, & $\Omega_2 = 1.5$. The known result is used to validate the outcome achieved in a certain model situation. The paper's system of equations is also concluded to offer useful information for comprehending the combined influence of temperature-dependent properties in multi-phase delays, as well as the thermal and diffusion characteristics of different bodies, in significant engineering challenges. These findings contribute to a deeper understanding of coupled thermoelastic diffusion phenomena and offer a foundation for future research on advanced thermoelastic models incorporating fractional-order derivatives, poroelastic effects, or memory-dependent materials.

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Availability of Data and Materials: Current submission does not contain the pool data of the manuscript but the data used in the manuscript will be provided on request.

Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest to report regarding the present study.

References

1. Fourier FMC. Theorie Analytique de la Chaleur. Paris, France: Firmin Didot; 1822. Oeuvres de Fourier.
2. Fick A. Uber diffusion. Ann Phys. 1855;94:59–86.
3. Lord HW, Shulman Y. A generalized dynamical theory of thermoelasticity. J Mech Phys Solid. 1967;15(5):299–309. doi:10.1016/0022-5096(67)90024-5.
4. Green AE, Lindsay KA. Thermoelasticity. J Elast. 1972;2(1):1–7. doi:10.1007/BF00045689.
5. Green AE, Naghdi PM. Thermoelasticity without energy dissipation. J Elast. 1993;31(3):189–208. doi:10.1007/BF00044969.
6. Hetnarski RB, Ignaczak J. Generalized thermoelasticity. J Therm Stress. 1999;22:451–76.
7. Tzou DY. Thermal shock phenomena under high rate response in solids. Annu Rev Heat Mass Treansfer. 1992;4(4):111–85. doi:10.1615/AnnualRevHeatTransfer.v4.50.
8. Cattaneo CA. A form of heat equation which eliminate the paradox of instantaneous propagation. CR Acad Sci. 1958;247:431–3.
9. Vernotte P. The paradoxes of the continuous theory of heat equation. Compt Rend. 1958;246:3154–5.
10. Tzou DY. A unified approach for heat conduction from macro-to-micro-scales. J Heat Transf. 1995;117(1):8–16. doi:10.1115/1.2822329.

11. Tzou DY. The generalized lagging response in small scale and high rate heating. Int J Heat Transf. 1996;38(17):3231–40. doi:10.1016/0017-9310(95)00052-B.
12. Roychouduri SK. On a thermoelastic three-phase-lags model. J Therm Stress. 2007;30(3):231–8. doi:10.1080/01495730601130919.
13. Tzou DY. Macro- to microscale heat transfer: the lagging behavior. NY, USA: John Wiley & Sons, Ltd.: 2014. doi:10.1002/9781118818275.ch2.
14. Nowacki W. Dynamical problems of thermodiffusion in solids. I. Bull Acad Pol Sci Ser Sci Tech. 1974;22: 55–64.
15. Nowacki W. Dynamical problems of thermodiffusion in solids. II. Bull Acad Pol Sci Ser Sci Tech. 1974;22:129–35.
16. Nowacki W. Dynamical problems of thermodiffusion in solids. III. Bull Acad Pol Sci Ser Sci Tech. 1974;22:257–66.
17. Nowacki W. Dynamical problems of thermodiffusion in elastic solids. Proc Vib Prob. 1974;15:105–28.
18. Sherief HH, Hamza F, Saleh HA. The theory of generalized thermoelastic diffusion. Int J Eng Sci. 2004;42:591–608. doi:10.1016/j.ijengsci.2003.05.001.
19. Kumar R, Kansal T. Propagation of lamb waves in transversely isotropic thermoelastic diffusive plate. Int J Solids Struct. 2008;45:5890–913. doi:10.1016/j.ijsolstr.2008.07.005.
20. Aouadi M. A theory of thermoelastic diffusion materials with voids. Z Angew Math Phys. 2010;61(2):357–79. doi:10.1007/s00033-009-0016-0.
21. Noda N. Thermal stresses in materials with temperature-dependent properties. Appl Mech Rev. 1991;44(9):383. doi:10.1115/1.3119511.
22. Ezzat MA, EI-Karamany AS, Samaan AA. The dependence of the modulus of elasticity on reference temperature in generalized thermoelasticity with thermal relaxation. Appl Math Comput. 2004;147(1):169–89. doi:10.1016/S0096-3003(02)00660-4.
23. Abouelregal AE. Fractional order generalized thermo-piezoelectric semi-infinite medium with temperature-dependent properties subjected to a ramp-type heating. J Therm Stress. 2011;34(11):1139–55. doi:10.1080/01495739.2011.606018.
24. Youssef HM, Ibrahim AA. Fractional order generalized thermoelasticity with variable thermal conductivity. J Vibroeng. 2014;16:4077–87.
25. Adel M, Lotfy K, Yadav AK, Ibrahim E. Orthotropic rotational semiconductor material with piezo-photothermal plasma waves with moisture plasma diffusion and laser pulse. Z Angew Math Mech. 2025;105(2):e202301004. doi:10.1002/zamm.202301004.
26. Aouadi M. Generalized Theory of thermoelastic diffusion for anisotropic media. J Therm Stress. 2008;31(3):270–85. doi:10.1080/01495730701876742.
27. Kuang ZB. Variational principles for generalized thermodiffusion theory in pyroelectricity. Acta Mech. 2010;214(3-4):275–89. doi:10.1007/s00707-010-0285-x.
28. Li Y, He TH, Tian XG. A generalized thermoelastic diffusion theory of porous materials considering strain, thermal and diffusion relaxation, and its application in the ultrashort pulse laser heating. Acta Mech. 2023;234(3):1083–103. doi:10.1007/s00707-022-03433-x.
29. Kumar R, Gupta V. Uniqueness, reciprocity theorem and plane waves in thermoelastic diffusion with fractional order derivative. Chin Phys B. 2013;22(7):1–12. doi:10.1088/1674-1056/22/7/074601.
30. Kumar R, Gupta V. Uniqueness, reciprocity theorems and plane wave propagation in different theories of thermoelasticity. Int J Appl Mech Eng. 2013;18(4):1067–86. doi:10.2478/ijame-2013-0067.
31. Sarkar I, Mukhopadhyay B. On energy, uniqueness theorems and variational principle for generalized thermoelasticity with memory-dependent derivative. Int J Heat Mass Transf. 2020;149(5):119112. doi:10.1016/j.ijheatmasstransfer.2019.119112.

32. Bala S, Khurana A, Tomar SK. Uniqueness, continuous and dependence and reciprocity theorems in the thermoelastic relaxed micromorphic continuum. J Therm Stress. 2021;44(6):715–30. doi:10.1080/01495739.2021.1893619.
33. Shivay ON, Mukhopadhyay S. Variational principle and reciprocity theorem on the temperature-rate-dependent poro-thermoelasticity theory. Acta Mech. 2021;232(9):3655–67. doi:10.1007/s00707-021-02996-5.
34. Al-Lehaibi EAN. The variationally principle of two-temperature thermoelasticity in the absence of the energy dissipation theorem. J Umm Al-Qura Univ Eng Arch. 2022;13:81–5. doi:10.1007/s43995-022-00003-z.
35. Paul K, Mukhopadhyay B. A novel mathematical model on generalized thermoelastic diffusion theory. J Therm Stress. 2023;46(4):253–75. doi:10.1080/01495739.2023.2176385.
36. Abouelregal AE. Modified fractional thermoelasticity model without energy dissipation including higher order application to spherical cavity exposed to a harmonic varying heat. Wave Random Complex Media. 2019;31(5):812–32. doi:10.1080/17455030.2019.1628320.
37. Abouelregal AE. Two-temperature thermoelastic model without energy dissipation including higher order time-derivatives and two phase-lags. Mater Res Express. 2019;6(11):116535. doi:10.1088/2053-1591/ab447f.
38. Abouelregal AE. On Green and Naghdi thermoelasticity model without energy dissipation with higher order time differential and phase-lags. J Appl Comput Mech. 2020;6(3):445–56.
39. Zenkour AM. Refined microtrmperatures multi-phase-lags theory for plane waves propagation in thermoelastic medium. Res Phys. 2018;11:929–37.
40. Zenkour AM. Refined multi-phase-lags theory for photothermal waves of a gravitational semiconducting half-space. Compos Struct. 2019;212(5):346–64. doi:10.1016/j.compstruct.2019.01.015.
41. Hafed Z, Zenkour A. Refined generalized theory for thermoelastic waves in a hollow sphere due to maintained constant temperature and radial stress. Case Stud Therm Eng. 2025;68(1):105905. doi:10.1016/j.csite.2025.105905.
42. Ezzat M, Fayik M. Fractional order theory of thermoelastic diffusion. J Therm Stress. 2011;34(8):851–72. doi:10.1080/01495739.2011.586274.
43. Kumar R, Sharma P. Variational principle, uniqueness and reciprocity theorems in porous piezothermoelastic with mass diffusion. J Solid Mech. 2016;8(2):446–65.
44. Alqahtani Z, Abbas I, El-Bary A. Piezothermoelastic interaction in piezoelectric rods induced by a timed laser pulse without energy dissipation. Case Stud Therm Eng. 2024;64(6):105470. doi:10.1016/j.csite.2024.105470.
45. Mondal N, Kashim Molla MA, Hossain Mallik S. A two-temperature generalized piezo-thermoelastic problem with Eringen's non-local effect and memory-dependent derivatives under three-phase-lag heat transfer. Waves Random Complex Media. 2022;65:1–23. doi:10.1080/17455030.2022.2161023.
46. Khamis AK, El-Bary AA, Lotfy K, Bakali A. Photothermal excitation processes with refined multi dual phase-lags theory for semiconductor elastic medium. Alex Eng J. 2020;59(1):1–9. doi:10.1016/j.aej.2019.11.016.
47. Lotfy K, El-Bary A. Magneto-photo-thermo-microstretch semiconductor elastic medium due to photothermal transport process. Silicon. 2022;14(9):4809–21. doi:10.1007/s12633-021-01205-1.
48. Sharma S, Sharma K, Bhargava RR. Wave motion and representation of fundamental solution in electro-microstretch viscoelastic solids. Mat Phys Mech. 2013;17(2):93–110.
49. Sharma K, Sharma S, Bhargava RR. Propagation of waves in micropolar thermoelastic solid with two temperatures bordered with layers or half spaces of inviscid liquid. Mat Phys Mech. 2013;16(1):66–81.
50. Kumar R, Devi S, Sharma V. Plane waves and fundamental solution in a modified couple stress generalized thermoelastic with mass diffusion. Mater Phys Mech. 2015;24(1):72–85.

51. Sharma S, Khator S. Power generation planning with reserve dispatch and weather uncertainties including penetration of renewable sources. Int J Smart Grid Clean Energy. 2021;10(4):292–303. doi:10.12720/sgce.10.4.292-303.
52. Sharma S, Khator S. Micro-grid planning with aggregator's role in the renewable inclusive prosumer market. J Power Energy Eng. 2022;10(4):47–62. doi:10.4236/jpee.2022.104004.
53. Hilal M. Diffusion, rotation and lagging behavior of a thermoelastic micropolar medium with voids and temperature gradient under mechanical pressure. Waves Random Complex Media. 2021;34(6):5702–21. doi:10.1080/17455030.2021.2011987.
54. Panja SK, Lahiri A, Mandal SC. A thermoelastic model with higher order time derivatives for a crack in a rotating solid. Int J Comput Methods Eng Sci Mech. 2022;24(2):107–18. doi:10.1080/15502287.2022.2066033.
55. Bibi F, Ali H, Azhar E, Jamal M, Ahmed I, Ragab AE. Propagation and reflection of thermoelastic waves in rotating nonlocal fractional order porous medium under hall current influence. Sci Rep. 2023;13(1):17703. doi:10.1038/s41598-023-44712-4.
56. El-Sapa S, El-Bary AA, Atef HM. The influence of rotation and viscosity on generalized conformable fractional micropolar thermoelasticity with two temperatures. Front Phys. 2023;11:1267808. doi:10.3389/fphy.2023.1267808.
57. Marin M, Marinescu C. Thermoelasticity of initially stressed bodies, asymptotic equipartition of energies Int J Eng Sci. 1998;36(1):73–86. doi:10.1016/S0020-7225(97)00019-0.
58. Marin M. Lagrange identity method for microstretch thermoelastic materials. J Math Anal Appl. 2010;363(1):275–86. doi:10.1016/j.jmaa.2009.08.045.
59. Marin M, Abbas I, Kumar R. Relaxed Saint-Venant principle for thermoelastic micropolar diffusion. Struct Eng Mech. 2014;51(4):651–62. doi:10.12989/sem.2014.51.4.651.
60. Abouelregal AE, Sedighi HM. A new insight into the interaction of thermoelasticity with mass diffusion for a half-space in the context of Moore–Gibson–Thompson thermodiffusion theory. Appl Phys A. 2021;127(8):582. doi:10.1007/s00339-021-04725-0.
61. Biot MA. Thermoelasticity and irreversible thermodynamic. J Appl Phys. 1956;27(3):240–53. doi:10.1063/1.1722351.
62. Zhang Z, Zhao W, Sun Y, Gu ZY, Qian WP, Gong H. Thermoelastic behaviors of temperature-dependent multilayer arches under thermomechanical loadings. Buildings. 2023;13(10):2607. doi:10.3390/buildings13102607.
63. Sherief HH, Saleh HA. A half-space problem in the theory of generalized thermoelastic diffusion. Int J Solids Struct. 2005;42(15):4484–93. doi:10.1016/j.ijsolstr.2005.01.001.