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EFFECT OF ROTATION OF THE BOUNDARIES ON THE STABILITY OF A FLOW CAUSED BY A NONLINEAR HEAT SOURCE ECCOMAS CONGRESS 2024

ARMANDS GRITSANS[1], ANDREI KOLYSHKIN[2], FELIX SADYRBAEV[3], AND INARA YERMACHENKO[1]

[1] Institute of Life Sciences and Technology
Daugavpils University
Parades street 1a, LV-5401, Daugavpils, Latvia
e-mail: armands.gricans@du.lv

[2] Institute of Applied Mathematics
Riga Technical University
Kipsalas street 6A, LV-1048, Riga, Latvia
email: andrejs.koliskins@rtu.lv

[3] Institute of Mathematics and Computer Science
University of Latvia
Raina boulevard 29, LV-1459, Riga, Latvia
email: felix@latnet.lv

[1]Institute of Life Sciences and Technology
Daugavpils University
Parades street 1a, LV-5401, Daugavpils, Latvia
e-mail: inara.jermacenko@du.lv

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Summary. Linear and weakly nonlinear instability of a convective flow in a tall vertical annulus is considered in the paper. The base flow has two components: (a) the vertical component generated by nonlinear heat sources and (b) the azimuthal component generated by rotation of the inner boundary. The properties of the nonlinear boundary value problem for the temperature distribution are analyzed in detail. It is proved, in particular, that there are two solutions of the boundary value problem for the base flow temperature distribution for a certain range of the Frank-Kamenetskii parameter. Linear stability problem is solved numerically. Calculations show destabilizing effect of rotation. Weakly nonlinear theory is used to construct the amplitude evolution equation for the most unstable mode. It is shown that the amplitude evolution equation is the complex Ginzburg-Landau equation.

1 INTRODUCTION

Biomass thermal conversion is one of the promising methods for green energy production [1]. Different factors such as convection, external electric field or degree of swirling may affect the efficiency of the conversion process [2], [3]. Mathematical modeling (in addition to experimental investigations) is an efficient tool for the analysis of complex multiphysics problems [4]. Stability theory can also be used (in combination with experimental and numerical analysis) in order to describe how and when a particular flow may become unstable [5]. In the context of biomass thermal conversion problem instability is desirable since it can lead to more intensive mixing and, as a result, to more efficient energy conversion.

In the present paper we analyze the combined effect of internal heat sources due to chemical reaction that takes place in the fluid, and rotation of the boundaries on the stability of a convective flow in a vertical annulus. Heat equation in this case has a nonlinear term so that the corresponding boundary value problem for the determination of the base flow temperature is nonlinear. Theoretical analysis of the nonlinear boundary value problem leads to the conclusion that in the region of interest in the parameter space for stability analysis there are two solutions of the corresponding boundary value problem. This fact is proved in the paper using the Krasnosels'kiĭ-Guo fixed point theorem (see [6]). The solution with the smallest norm should be chosen as the base flow temperature distribution since the other solution is unstable and cannot be observed in experiments. Linear stability of the base flow is performed numerically. It is shown that rotation of the boundaries destabilizes the flow. Assuming that the parameters of the problem are chosen in a small neighborhood of the critical point where the flow is linearly unstable with very small growth rate, we derive the amplitude evolution equation for the most unstable mode using the method of multiple scales. It is shown that the corresponding amplitude evolution equation is the complex Ginzburg-Landau equation.

2 MATHEMATICAL FORMULATION OF THE PROBLEM

Consider the flow of a viscous incompressible fluid in the domain $D = \{R_1 < \tilde{r} < R_2, 0 \leq \varphi < 2\pi, -\infty < \tilde{z} < +\infty\}$ between two infinitely long concentric cylinders with radii R_1 and R_2, respectively ($R_1 < R_2$). The walls of the cylinders are maintained at constant equal temperatures $\tilde{\theta}_0$. It is also assumed that the inner cylinder is rotating with constant angular velocity ω while the outer cylinder is at rest. We use the system of cylindrical polar coordinates $(\tilde{r}, \varphi, \tilde{z})$ with the origin at the axes of the cylinders. The flow is described by the system of the Navier-Stokes equations under the Boussinesq approximation [7] written in dimensionless form as follows:

$$\frac{\partial \mathbf{v}}{\partial t} + Gr(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \Delta\mathbf{v} + T\mathbf{e}_k, \quad (1)$$

$$\frac{\partial T}{\partial t} + Gr\mathbf{v}\cdot\nabla T = \frac{1}{Pr}\Delta T + \frac{Q}{Pr}, \quad (2)$$

$$\nabla\cdot\mathbf{v} = 0, \quad (3)$$

where $\mathbf{v}$ is the velocity, T is the temperature, p is the pressure, Q is the density of the internal heat sources, and $\mathbf{e}_k = (0, 0, 1)$. The flow is characterized by two dimensionless parameters, namely, the Prandtl number Pr and the Grashof number Gr (see [8] for details). Different forms of the function Q are considered in the literature: (a) the case $Q = \text{const}$ corresponds to Joule heating of an electrolyte, (b) the case $Q = Q_0 \exp{(az)}$ corresponds to the situation where a light beam is passing through the fluid in such a way that the intensity of light is decreasing exponentially (Bouguer-Beer-Lambert law, [9]), (c) the case

$$Q = \tilde{Q}_0 \exp{[-E/(R_0\tilde{T})]}, \quad (4)$$

corresponds to the Arrhenius' law where E is the activation energy, R_0 is the universal gas constant and $\tilde{T}$ is the absolute temperature. Formula 4 is used to describe the thermal effect of a chemical reaction that takes place in the fluid. The following convention is used throughout the paper: all the variables with tildes are dimensional while the variables without tildes are dimensionless.

There exists a steady solution of 1–3 of the following form:

$$\mathbf{v}_0 = (0, V_0(r), W_0(r)), \quad T = T_0(r), \quad p = p_0(z). \tag{5}$$

Substituting (5) into (1)–(3) we obtain the following system of ordinary differential equations

$$W_0^{''} + \frac{W_0^{'}}{r} + T_0 = C, \tag{6}$$

$$V_0^{''} + \frac{V_0^{'}}{r} - \frac{V_0}{r^2} = 0, \tag{7}$$

$$T_0^{''} + \frac{T_0^{'}}{r} + Fe^{T_0} = 0, \tag{8}$$

where $C = dp_0/dz$. Note that the Frank-Kamenetskii transformation is used in order to simplify the source term in (2) and (4). The idea is rather simple: to expand the exponent in (4) in a Taylor series and keep only the linear terms of the series. The result is mathematically more convenient term to work with (see [10]). The accuracy of the transformation is analyzed in [10] and [11] where it is shown that it is rather accurate in a wide range of the parameters of interest.

The boundary conditions are

$$V_0(R) = S, \quad V_0(1) = 0, \tag{9}$$

$$W_0(R) = 0, \quad W_0(1) = 0, \tag{10}$$

$$T_0(R) = 0, \quad T_0(1) = 0, \tag{11}$$

where $R = R_1/R_2$ and $S = \omega R_1/u_0$ (the scaling parameter u_0 is defined in [13]). The annulus is assumed to be closed (at infinity) so that the total fluid flux through the cross-section is equal to zero:

$$\int_R^1 W_0(r) r \, dr = 0. \tag{12}$$

It follows from (7), (9) that the boundary value problem for the function $V_0(r)$ can be solved separately. The solution has the form

$$V_0(r) = -\frac{SR}{1-R^2} r + \frac{SR}{r(1-R^2)}. \tag{13}$$

The boundary value problem for the function $T_0(r)$ also can be solved separately (in addition, only this problem has a nonlinear term). The analysis of the number of solutions of the boundary value problem (8), (11) is performed in the next section.

3 NONLINEAR BOUNDARY VALUE PROBLEM

We consider a generalization of (8), (11) in the form

$$x'' + \frac{1-\alpha}{r}x' + Fe^{x} = 0, \quad x(R) = 0 = x(1), \tag{14}$$

where $\alpha \in \mathbb{R}$, $F \in \mathbb{R}_{+}$, $0 < R < 1$. It can be shown (see [8]) that a nonzero α in (14) corresponds to the radial Reynolds number for the case where there is a radial inflow ($\alpha < 0$) or outflow ($\alpha > 0$) through the permeable walls of the cylinders. We observed the solvability and multiplicity of positive solutions $x(r)$ of (14) simultaneously. If $x(r)$ is a positive solution of (14), then $x'(R) > 0$ and $x'(1) < 0$. The approach is the same as in work [16], based on application of the Krasnosels'kiĭ-Guo fixed point theorem of cone expansion and compression of norm type.

For all values of α, the linear homogeneous problem

$$x'' + \frac{1-\alpha}{r}x' = 0, \quad x(R) = 0 = x(1) \tag{15}$$

has only the trivial solution and thus there exists a unique Green's function, $G(r,s)$, related to (15).

$$\text{If} \quad \alpha = 0, \quad \text{then} \quad G_1(r,s) = \begin{cases} \dfrac{s\ln r(\ln R - \ln s)}{\ln R}, & \text{if} \quad R \le s \le r \le 1, \\ \dfrac{s\ln s(\ln R - \ln r)}{\ln R}, & \text{if} \quad R \le r < s \le 1. \end{cases} \tag{16}$$

$$\text{If} \quad \alpha \neq 0, \quad \text{then} \quad G_2(r,s) = \begin{cases} \dfrac{(r^{\alpha}-1)(s^{\alpha}-R^{\alpha})}{\alpha s^{\alpha-1}(1-R^{\alpha})}, & \text{if} \quad R \le s \le r \le 1, \\ \dfrac{(s^{\alpha}-1)(r^{\alpha}-R^{\alpha})}{\alpha s^{\alpha-1}(1-R^{\alpha})}, & \text{if} \quad R \le r < s \le 1. \end{cases} \tag{17}$$

$G_i(r,s)$, $(i \in \{1,2\})$ is defined in the square $\Omega := \big\{(r,s) \in \mathbb{R}^2 : \ R \le r \le 1, \ R \le s \le 1\big\}$. Denote by $\overset{\circ}{\Omega}$ and $\partial\Omega$ the interior and the boundary of Ω, respectively. Let us introduce the function $k_i : \Omega \to \Omega$, $i \in \{1,2\}$

$$k_i(r,s) := -G_i(r,s), \quad (r,s) \in \Omega,$$

and the function $\Phi_i : [R,1] \to [R,1]$, $i \in \{1,2\}$

$$\Phi_i(s) := k_i(s,s), \quad s \in [R,\, 1].$$

If $\alpha = 0$, then we have

$$\Phi_1(s) := \frac{s\ln s(\ln s - \ln R)}{\ln R}. \tag{18}$$

If $\alpha \neq 0$, then we obtain

$$\Phi_2(s) := \frac{(1-s^{\alpha})(s^{\alpha}-R^{\alpha})}{\alpha s^{\alpha-1}(1-R^{\alpha})}. \tag{19}$$

Remark. The function $\Phi_2(s)$ is symmetric with respect to α. $\Phi_2(s;-\alpha) = \Phi_2(s;\alpha)$.

Proposition 1. The functions k_i and Φ_i have the following properties.

- $k_i(r,s) > 0$ for every $(r,s) \in \overset{\circ}{\Omega}$ and $k_i(r,s) = 0$ for every $(r,s) \in \partial\Omega$.
- $\Phi_i(s) > 0$ for every $s \in (R,1)$ and $\Phi_i(R) = 0 = \Phi_i(1)$.
- $\Phi_i(s) \geq k_i(r,s)$ for every $r, s \in [R,1]$.
- Let a and b be two positive numbers such that $R < a < b < 1$.
  Let $c_i := \min\{c_{i,1};\, c_{i,2}\}$, $i \in \{1,2\}$, where

$$c_{1,1} := \frac{\ln b}{\ln R}, \qquad c_{1,2} := 1 - \frac{\ln a}{\ln R}, \qquad \text{if} \quad \alpha = 0; \tag{20}$$

$$c_{2,1} := \frac{1 - b^{\alpha}}{1 - R^{\alpha}}, \quad c_{2,2} := \frac{a^{\alpha} - R^{\alpha}}{1 - R^{\alpha}}, \qquad \text{if} \quad \alpha \neq 0. \tag{21}$$

  Then, $c_i \in (0,1)$ and $c_i\, \Phi_i(s) \leq k_i(r,s)$ for $\forall r \in [a,b]$ and $\forall s \in [R,1]$.

Proof. The assertions of Proposition 1 are valid by straightforward calculations.□

We consider the Banach space $C_{[R,1]}$ with the norm $\|x\| := \max\limits_{R \leq r \leq 1} \big|x(r)\big|$. Define an integral operator $T : C_{[R,1]} \to C_{[R,1]}$,

$$Tx(r) := F \int_R^1 k(r,s) e^{x(s)} ds, \quad x \in C_{[R,1]}, \quad r \in [R,1], \tag{22}$$

where $k(r,s) = k_1(r,s)$, if $\alpha = 0$, or $k(r,s) = k_2(r,s)$, if $\alpha \neq 0$.
A function $x(r)$ is a solution of the boundary value problem (14) if and only if $x(r)$ is a solution of the integral equation

$$x(r) = F \int_R^1 k(r,s) e^{x(s)} ds, \quad r \in [R,1], \tag{23}$$

thereby the fixed points of T coincide with the solutions of (14).
Let a and b be two real numbers such that $R < a < b < 1$. Let c be constant defined by (20), (21). Just like in paper [16] we consider the cones in the Banach space $C_{[R,1]}$

$$P := \big\{ x \in C_{[R,1]} :\ x(r) \geq 0,\ r \in [R,1] \big\},$$
$$K := \Big\{ x \in P :\ \min_{r \in [a,b]} x(r) \geq c\, \|x\| \Big\},$$

and prove the following properties of the integral operator T.
Proposition 2. The operator T defined by (22) has the following properties.

- For every $x \in C_{[R,1]}$, $Tx(r) > 0$ for all $r \in (R,1)$ and $Tx(R) = 0 = Tx(1)$.
- $T\left(C_{[R,1]}\right) \subset K$.
- $T(P) \subset P$ and the operator $T : P \to P$ is completely continuous.
- $T(K) \subset K$ and the operator $T : K \to K$ is completely continuous.

We use the Krasnosels'kiĭ-Guo theorem; see ([6], Theorem 2.3.4), ([12], Theorem 1.0.3).

Theorem 1. Let E be a Banach space and let M be a cone in E. Let $T : M \to M$ be a completely continuous operator. Assume that there exist two positive constants $\rho, \overline{\rho}$ with $\rho < \overline{\rho}$ such that one of the following conditions:

(H1) $\|Tx\| \le \|x\|$ for every $x \in M$ with $\|x\| = \rho$ and $\|Tx\| \ge \|x\|$ for every $x \in M$ with $\|x\| = \overline{\rho}$,

(H2) $\|Tx\| \ge \|x\|$ for every $x \in M$ with $\|x\| = \rho$ and $\|Tx\| \le \|x\|$ for every $x \in M$ with $\|x\| = \overline{\rho}$,

is satisfied. Then T has a fixed point x in M such that $\rho \le \|x\| \le \overline{\rho}$.

Let ρ be a positive number. Let a and b be two real numbers such that $R < a < b < 1$. Let c be the constant defined by (20) or (21). Let us introduce two functions:

$$F^*(\rho) := \frac{\rho}{e^{\rho} \int_R^1 \Phi(s)ds}; \tag{24}$$

$$F_*(\rho) := \frac{\rho}{ce^{c\rho} \int_a^b \Phi(s)ds}, \tag{25}$$

where $\Phi(s)$ is defined by (18) or (19).

Lemma 1. The following assertions are fulfilled.

1. The function $F^* : (0, +\infty) \to \mathbb{R}_+$ defined in (24) has the following properties
   - $\lim\limits_{\rho \to 0+} F^*(\rho) = 0 = \lim\limits_{\rho \to +\infty} F^*(\rho)$;
   - strictly increases in $(0, 1]$, strictly decreases in $[1, +\infty)$, and has a unique global maximum point $\rho = 1$;
2. The function $F_* : (0, +\infty) \to \mathbb{R}_+$ defined in (25) has the following properties
   - $\lim\limits_{\rho \to 0+} F_*(\rho) = 0 = \lim\limits_{\rho \to +\infty} F_*(\rho)$;
   - strictly increases in $(0, 1/c]$, strictly decreases in $[1/c, +\infty)$, and has a unique global maximum point $\rho = 1/c$;
3. $F^*(\rho) < F_*(\rho)$ for every positive ρ.

Proof. Lemma 1 can be proved by elementary calculus.□

Lemma 2. If $F \le F^*(\rho)$, then $\|Tx\| \le \|x\|$ for every $x \in P$ with $\|x\| = \rho$.

Proof. By Proposition 1, we have $\int_R^1 \Phi(s)ds > 0$. Namely, taking into account (18) and (19), we have

$$\text{if} \quad \alpha = 0, \quad \text{then} \quad \int_R^1 \Phi(s)ds = \frac{R^2 + 1}{4} + \frac{1 - R^2}{4 \ln R}; \tag{26}$$

$$\text{if} \quad \alpha \ne 0, \alpha \ne \pm 2 \quad \text{then} \quad \int_R^1 \Phi(s)ds = \frac{2(R^2+1)(1-R^{\alpha}) - \alpha(R^{\alpha}+1)(1-R^2)}{2(4-\alpha^2)(1-R^{\alpha})}; \tag{27}$$

$$\text{if} \quad \alpha = \pm 2, \quad \text{then} \quad \int_R^1 \Phi(s)ds = \frac{R^2 \ln R}{2(1-R^2)} + \frac{R^2+1}{8}. \tag{28}$$

Therefore, $F^*(\rho)$ in (24) is well-defined.

Let x be an element of P such that $\|x\| = \rho$. We see that $x(s) \le \rho$ for every $s \in [R, 1]$ and thus $e^{x(s)} \le e^{\rho}$ for every $s \in [R, 1]$. For an arbitrary $r \in [R, 1]$, we have

$$Tx(r) \le Fe^{\rho} \int_R^1 \Phi(s)ds \le \rho = \|x\|.$$

In view of Proposition 2, we obtain $\|Tx\| \le \|x\|$.□

Lemma 3. If $F \ge F_*(\rho)$, then $\|Tx\| \ge \|x\|$ for every $x \in K$ with $\|x\| = \rho$.

Proof. By Proposition 1, $F_*(\rho)$ in (25) is well-defined since $\int_a^b \Phi(s)ds > 0$. Let x be an element of K such that $\|x\| = \rho$. Hence, $x(s) \ge \min_{s\in[a,b]} x(s) \ge c\|x\| = c\rho$ for every $s \in [a, b]$ and thus $e^{x(s)} \ge e^{c\rho}$ for every $s \in [a, b]$. For an arbitrary $r \in [a, b]$, we have

$$Tx(r) \ge Fc \int_R^1 \Phi(s)e^{x(s)}ds \ge Fc \int_a^b \Phi(s)e^{x(s)}ds \ge Fce^{c\rho} \int_a^b \Phi(s)ds \ge \rho = \|x\|.$$

Therefore, $\max_{r\in[R,1]} Tx(r) \ge \|x\|$. In view of Proposition 2, we obtain $\|Tx\| \ge \|x\|$. □

Theorem 2.(Main result) Suppose that α is a real number and $R \in (0, 1)$. If a positive F satisfies $F < F^*(1)$, then the problem (14) has two positive solutions x_1 and x_2 such that $\|x_1\| < 1 < \|x_2\|$.

Proof. Let c be the constant defined by (20) or (21).

(1) It follows from Lemma 1 that there exists a unique $\overline{\rho}_1$ in the interval $(0, 1)$ such that $F = F^*(\overline{\rho}_1)$. On account of Lemma 2, $\|Tx\| \le \|x\|$ for every $x \in K$ with $\|x\| = \overline{\rho}_1$. By Proposition 1 and Lemma 1, we have $1 < 1/c$ and $F^*(1) < F_*(1) < F_*(1/c)$. It follows from Lemma 1 that there exists a unique ρ_1 in the interval $(0, 1)$ such that $F = F_*(\rho_1)$. On account of Lemma 3, $\|Tx\| \ge \|x\|$ for every $x \in K$ with $\|x\| = \rho_1$. In view of Lemma 1, $\rho_1 < \overline{\rho}_1$. Thereby, by Theorem 1(H2), the operator T has a fixed point x_1 in K such that $\rho_1 \le \|x\| \le \overline{\rho}_1$.

(2) It follows from Lemma 1 that there exists a unique ρ_2 in the interval $(1, +\infty)$ such that $F = F^*(\rho_2)$. On account of Lemma 2, $\|Tx\| \le \|x\|$ for every $x \in K$ with $\|x\| = \rho_2$. Since $F^*(1) < F_*(1/c)$, it follows from Lemma 1 that there exists a unique $\overline{\rho}_2$ in the interval $(1/c, +\infty)$ such that $F = F_*(\overline{\rho}_2)$. On account of Lemma 3, $\|Tx\| \ge \|x\|$ for every $x \in K$ with $\|x\| = \overline{\rho}_2$. In view of Lemma 1, $\rho_2 < \overline{\rho}_2$. Thereby, by Theorem 1(H1), the operator T has a fixed point x_2 in K such that $\rho_2 \le \|x_2\| \le \overline{\rho}_2$.

We note that $\overline{\rho}_1 < 1 < \rho_2$. The fixed points of T coincide with the positive solutions of (14) and thus it follows from (1) and (2) that (14) has two positive solutions x_1 and x_2 such that $\|x_1\| < 1 < \|x_2\|$.□

4 LINEAR STABILITY ANALYSIS

Consider a perturbed flow of the form

$$\mathbf{v} = \mathbf{v_0} + \mathbf{v}', \quad T = T_0 + T', \quad p = p_0 + p', \tag{29}$$

where $\mathbf{v}', T', p'$ are small unsteady perturbations, which are assumed in the form of normal modes

$$\begin{aligned} \mathbf{v}'(r,z,t) &= \mathbf{u}(r)\, e^{-\lambda t + ikz}, \\ T'(r,z,t) &= \theta(r)\, e^{-\lambda t + ikz}, \\ p'(r,z,t) &= q(r)\, e^{\lambda t + ikz}, \end{aligned} \tag{30}$$

where $\mathbf{u}(r) = (u(r), v(r), w(r))$, k is the wave number and $\lambda = \lambda_r + i\lambda_i$ is a complex eigenvalue. Note that only axisymmetric perturbations are considered in the paper. The reason is our choice of R. It is known from [14] that asymmetric perturbations are the most unstable for wide gaps ($R \leq 0.3$) while for $R = 0.7$ the axisymmetric perturbations are the most unstable.

As it follows from the results of the previous section, there are two solutions of the boundary-value problem (6)–(11) in the interval $0 < F < F^*(1)$. The solution with the smallest norm should be chosen since the other solution is unstable with respect to small perturbations (and, therefore, cannot be observed in experiments). The base flow velocity and temperature distributions (corresponding to the temperature distribution with the smallest norm) are obtained numerically using (6), (8), (10), (11, and (12). The corresponding graphs are shown in Figures 1 and 2.

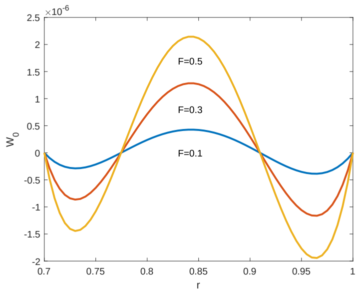

Figure 1: Vertical component of the base flow velocity distribution for three values of F.

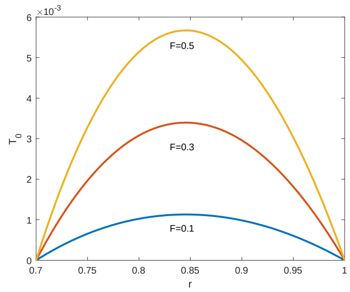

Figure 2: Base flow temperature distribution for three values of F.

Substituting (30) into (1)–(3) and linearizing the resulting equations in the neighborhood of the base flow we obtain the system of ordinary differential equations (with zero boundary conditions) which is solved numerically using the collocation method (the details of the numerical procedure are described in [8]). Marginal stability curves for $F = 0.1$ and three values of S, namely, $S = 0.5, 0.9$ and 1.6 are shown in Figure 3. As can be seen from the figure, rotation of the inner boundary destabilizes the flow (the flow is linearly stable below the marginal stability curve and linearly unstable above it).

Figure 4 plots the critical Grashof numbers versus S for one value of the parameter F, namely, for $F = 0.1$. Destabilizing effect of rotation is clearly seen from Figure 4.

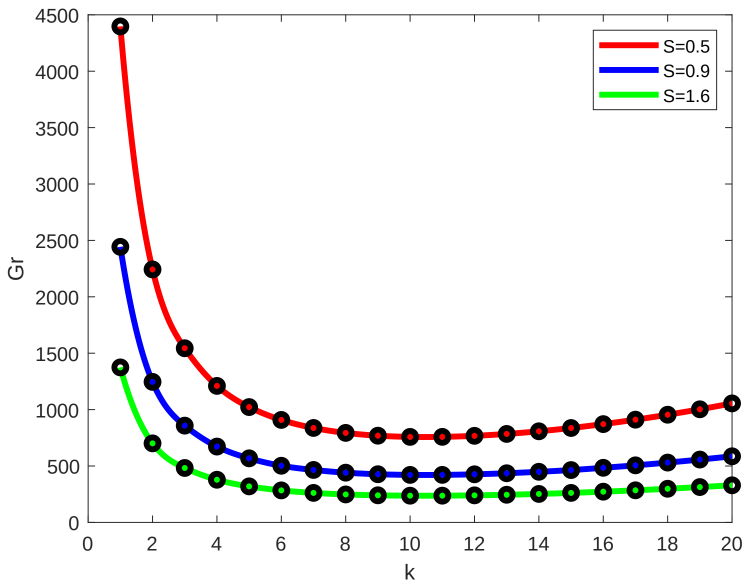

Figure 3: Marginal stability curves for three values of S and $F = 0.1$.

5 WEAKLY NONLINEAR ANALYSIS

In order to analyze the behavior of the most unstable mode in the neighborhood of the critical point (assuming that the Grashof number is slightly above the critical value, namely, $Gr = Gr_c(1+\varepsilon^2)$) weakly nonlinear theory can be applied. In this section we briefly describe the procedure based on the weakly nonlinear approach. Let $\mathbf{h} = (u, v, w, T, p)^T$ be the vector containing the unknown functions. We expand $\mathbf{h}$ in a power series in ε:

$$\mathbf{h} = \mathbf{h_0} + \varepsilon \mathbf{h_1} + \varepsilon^2 \mathbf{h_2} + \varepsilon^3 \mathbf{h_3} + ... \tag{31}$$

We introduce "slow" variables $\xi = \varepsilon(z - c_g t)$ and $\tau = \varepsilon^2 t$ (where c_g is the group velocity). Substituting (31) into (1)–(3) we obtain the system of equations at orders ε, ε^2 and ε^3 of the form

$$L\mathbf{h_1} = 0, \tag{32}$$
$$L\mathbf{h_2} = \mathbf{f_1}, \tag{33}$$
$$L\mathbf{h_3} = \mathbf{f_2}, \tag{34}$$

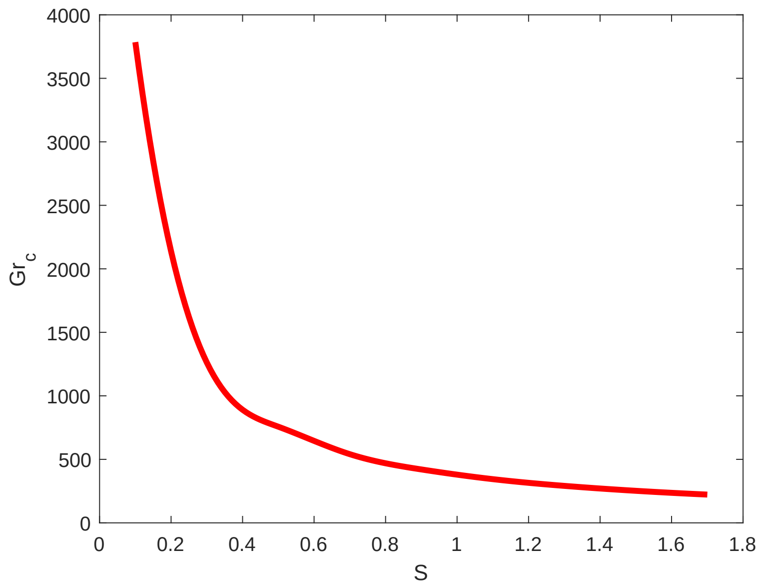

Figure 4: Critical values of the Grashof number versus S for $F = 0.1$.

where the operator L is defined by (1)–(3) with the corresponding zero boundary conditions. Using the method of normal modes we assume perturbations of the form

$$\mathbf{h} = \mathbf{g_1}(r) \exp ik(z - ct), \tag{35}$$

where $\mathbf{g_1}(r)$ is the eigenvector representing the amplitudes of the normal perturbations, k is the wave number and c is the wave speed of the perturbation. Note that in accordance with the linear stability theory the eigenvector cannot be uniquely defined. Thus, we modify (35) and introduce the amplitude function $A = A(\xi, \tau)$ so that the perturbations are assumed of the form

$$\mathbf{h_1} = A(\xi, \tau)\mathbf{g_1}(r) \exp ik(z - ct) + c.c., \tag{36}$$

where the abbreviation c.c. means "complex conjugate". Substituting (36) into (32)–(34) we obtain

$$L_1\mathbf{g_1} = 0, \tag{37}$$
$$L_1\mathbf{g_2} = \mathbf{s_1}, \tag{38}$$
$$L_1\mathbf{g_3} = \mathbf{s_2}. \tag{39}$$

The function $\mathbf{f_1}$ in (33) contains the terms proportional to AA^*, A_ξ and A^2. Thus, the solution to (33) can be represented in the form

$$\mathbf{h_2} = AA^*\mathbf{g_2}^{(0)}(r) + A_\xi\mathbf{g_2}^{(1)}(r)\, e^{ik(z-ct)} + A^2\mathbf{g_2}^{(2)}(r)\, e^{2ik(z-ct)} + c.c. \tag{40}$$

The functions $\mathbf{g_2}(r)^{(i)}, i = 0, 1, 2$ can be found from the solutions of three boundary value problems that are obtained by substituting (40) into (33). It can be shown that the solution to (38) exists if and only if the function $\mathbf{s_1}$ in (38) is orthogonal to all eigenfunctions of the corresponding adjoint problem (see [15]). This solvability condition follows from the Fredholm's Alternative and allows one to determine the group velocity c_g. Next, we substitute (36) and (40) into (34) and apply the Fredholm's Alternative to (39). The result is the complex Ginzburg-Landau equation of the form

$$A_\tau = \sigma A + \delta A_{\xi\xi} - \mu|A|^2 A. \tag{41}$$

The coefficients of the equations are obtained in terms of integrals containing the eigenfunction of the linear stability problem, the eigenfunction of the corresponding adjoint problem, and the functions $\mathbf{g_2}(r)^{(i)}, i = 0, 1, 2$.

6 CONCLUSIONS

In the present paper we analyze linear and weakly nonlinear instability of a convective flow in a tall vertical annulus. The base flow component in the vertical direction is generated due to nonlinear heat sources (heat is released in the fluid as a result of a chemical reaction). It is assumed that the inner cylinder is rotating with constant angular velocity while the outer cylinder is at rest. The nonlinear boundary value problem for the base flow temperature distribution is analyzed in detail. It is proved in the paper that for a certain range of the Frank-Kamenetskii parameter F there exist two solutions of the corresponding boundary value problem such that the norm of one solution is smaller than one and the norm of the other solution is larger than one. The solution with the smallest norm is chosen for linear stability analysis since the other solution is unstable. Linear stability calculations show destabilizing effect of rotation. Multiple scale expansion in the neighborhood of the critical point results in evolution equation for the amplitude of the most unstable mode. It is shown that the corresponding equation is the complex Ginzburg-Landau equation.

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