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TIME-DEPENDENT UNCERTAINTY QUANTIFICATION ANALYSIS OF COMPLEX DYNAMICAL SYSTEMS

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Abstract. This paper presents a novel computational method, accompanied by robust numerical algorithms, for time-dependent uncertainty quantification (UQ) analysis of complex dynamical systems. The method involves a stochastic adaptation of the nonlinear auto regressive with exogenous input (NARX) model to accurately capture the underlying dynamical system behavior, a polynomial dimensional decomposition (PDD) of random coefficients generated by NARX, and a new integration between NARX and PDD, establishing the PDD-NARX approximation. The PDD-NARX method proposed is distinguished from conventional deterministic system identification tools by accounting for uncertainties arising both from the dynamic system properties (e.g., mass, stiffness, damping) and from the external forces (e.g., amplitude and frequency content of excitation time series). In addition, PDD, owing to its hierarchical, dimensionwise expansion, is capable of handling high-dimensional UQ problems better than many existing methods, including polynomial chaos expansion. Numerical results demonstrate that low-order PDD-NARX approximations not only provide accurate and computationally efficient estimates of the probabilistic characteristics of simple dynamical systems, but also underscore the method's effectiveness in addressing industrial-scale complex problems, as evidenced by the probabilistic vehicle dynamic analysis of a pick-up truck traversing road bumps.

1 INTRODUCTION

Mathematical modeling and simulation of complex dynamical systems often require uncertainty quantification (UQ) due to inherent randomness in system properties, external excitations, and boundary/initial conditions [1, 2]. Uncertainty propagation is commonly assessed using sampling-based methods like Monte Carlo simulation (MCS), which, despite being robust, becomes impractical for computationally expensive dynamic analyses.

To address this, surrogate methods such as polynomial chaos expansion (PCE) [3], polynomial dimensional decomposition (PDD) [4, 5], stochastic collocation [6], and sparse-grid quadrature [7] are frequently employed. While effective for time-independent or quasi-static problems, these approaches struggle with time-dependent dynamics, where maintaining accuracy over long-time integration often demands prohibitively high-order expansions.

Gerritsma et al. [8] proposed a time-dependent PCE to improve long-time integration, but challenges remain for more complex systems. For such problems, system identification techniques like the nonlinear autoregressive with exogenous input (NARX) model offer a viable alternative. Hybrid approaches, such as PCE-NARX, have shown promise for time-dependent UQ [9, 10], though they still face two major limitations.

First, the selection of an appropriate basis for the NARX model remains largely intuitive and can be ambiguous, as multiple options exist. Second, in high-dimensional UQ problems commonly encountered in industrial-scale applications, the number of basis functions or expansion coefficients in PCE grows exponentially, thus succumbing to the curse of dimensionality.

The primary objective of this study is to introduce a novel computational method, referred to as the PDD-NARX approximation, along with its associated numerical algorithms, for time-dependent UQ analysis of complex dynamical systems. Section 2 presents the definition and setup of the time-dependent UQ problem, along with necessary assumptions. In Section 3, the stochastic adaptation of the NARX model is described, utilizing dimensional decomposition followed by basis reduction via a four-step algorithm. Section 4 introduces the PDD approximation of the random NARX coefficients. In Section 5, the integration between PDD and NARX is explained, establishing the PDD-NARX approximation. Section 6 addresses a large-scale engineering problem, where the probabilistic vehicle dynamic analysis of a pickup truck is conducted, demonstrating the practical applicability of the PDD-NARX method. Section 7 provides conclusions drawn from the study.

2 Input Random Variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is a sample space, $\mathcal{F}$ is a σ-field on Ω, and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure. With $\mathcal{B}^N$ representing the Borel σ-field on $\mathbb{A}^N$, $N \in \mathbb{N} := \{1, 2, \cdots\}$, consider an $\mathbb{R}^N$-valued continuous random vector $\mathbf{X} := \{X_1, \cdots, X_N\}^\intercal : (\Omega, \mathcal{F}) \to (\mathbb{A}^N, \mathcal{B}^N)$, which describes the statistical uncertainties in all input parameters of a stochastic-dynamics or time-dependent UQ problem. Denote by $f_{\mathbf{X}}(\mathbf{x})$ the joint probability density function (PDF) of $\mathbf{X}$.

The requisite assumptions on input random variables are as follows: 1. All component random variables X_i, $i = 1, \ldots, N$, are statistically independent, but not necessarily identically distributed. 2. Each input random variable X_i, defined on a bounded or unbounded interval $\mathbb{A} := (a_k, b_k)$ of $\mathbb{R}$, has finite moments of all orders. 3. Each input random variable X_i has continuous marginal PDF $f_{X_i}(x_i)$ with a bounded or unbounded support $\mathbb{A}$. Moreover, given an infinite sequence of moments of X_i the PDF $f_{X_i}(x_i)$ is uniquely determined.

These assumptions are frequently adopted by the UQ community.

2.1 Time-Dependent UQ Problem

Let $\mathbf{y}(t; \mathbf{X}) : [0, T] \times \mathbb{A}^N \to \mathbb{R}^K$, $T \in \mathbb{R}^+$, $K \in \mathbb{N}$, be a general K-dimensional vector-valued stochastic dynamic response of interest at time $t \in [0, T]$ and $L^2(\Omega, \mathcal{F}, P)$ a Hilbert space of square-integrable functions $\mathbf{y}$ with respect to $f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$. The arguments of the output function $\mathbf{y}$ indicate that it depends not only on time t, but also on the input random vector $\mathbf{X}$. For a general time-dependent UQ problem, $\mathbf{y}(t; \mathbf{X})$ satisfies P-almost surely the parameterized stochastic differential equations (SDEs)

$$\begin{aligned} \mathcal{A}\left[\mathbf{y}(t; \mathbf{X})\right] &= g(t; \mathbf{X}), \quad t \in [0, T] \subseteq \mathbb{R}_0^+, \, \mathbf{y} \in L^2(\Omega, \mathcal{F}, P), \\ \mathcal{C}\left[\mathbf{y}(0; \mathbf{X})\right] &= q(\mathbf{X}), \end{aligned} \tag{1}$$

where $\mathcal{A}$ is a linear or nonlinear differential operator describing dynamics of discrete systems, $\mathcal{C}$ is an initial condition operator, and g and q, possibly random, are the forcing term and initial condition, respectively and consider $y(t;\mathbf{X})$ to be any component of $\mathbf{y}(t;\mathbf{X})$. In this study, the authors focus on calculating the second-moment statistics of $y(t;\mathbf{X})$ for complex dynamical systems.

In this paper, the proposed PDD-NARX approximation is described in the context of solving a general time-dependent UQ problem represented by Eq. (1). The exposition involves (1) a stochastic adaptation of NARX to accurately capture the underlying dynamical system behavior; (2) a PDD approximation of random coefficients generated by NARX; and (3) a new integration between NARX and PDD, establishing the PDD-NARX approximation.

3 Stochastic NARX

Consider a general system identification problem aimed at building a mathematical model to describe approximately the output response $y(t;\mathbf{X})$ of a time-dependent or dynamic system subjected to the input excitation $g(t;\mathbf{X})$. Using the observed or calculated data of the input and output signals, such approximation allows one to determine the output function $y(t;\mathbf{X})$ without directly solving Eq. (1) at a time t of interest.

3.1 NARX Approximation

Given a time interval $[0,T]$, $T\in[0,\infty)$, and a chosen integer $J\in\mathbb{N}$, let

$$0=t_0<t_1<\cdots<t_J=T<\infty,\; t_j\in[0,T],\; j=0,1,\ldots,J,$$

be $(J+1)$ discrete time instants with the constant time step $\Delta t=t_j-t_{j-1}, j=1,\ldots,J$. The uniform spacing of discrete times is merely for simplicity, as variable time steps can be handled rather easily. According to NARX [10, 11], the dynamic response $y(t_j;\mathbf{X})$ at a current time t_j is approximated by

$$\tilde{y}(t_j;\mathbf{X})=F\left(g(t_j;\mathbf{X}),g(t_j-\Delta t;\mathbf{X}),\ldots,g(t_j-n_g\Delta t;\mathbf{X}),y(t_j-\Delta t;\mathbf{X}),\ldots,y(t_j-n_y\Delta t;\mathbf{X})\right), \quad (2)$$

where,

$$\mathbf{z}(t_j;\mathbf{X}):=\{g(t_j;\mathbf{X}),g(t_j-\Delta t;\mathbf{X}),\ldots,g(t_j-n_g\Delta t;\mathbf{X}),y(t_j-\Delta t;\mathbf{X}),\ldots,y(t_j-n_y\Delta t;\mathbf{X})\}^{\intercal}, \quad (3)$$

and $n_g\in\mathbb{N}$, $n_y\in\mathbb{N}$ are time lags and F is the NARX model function.

It is obvious that the approximation quality of $\tilde{y}(t_j;\mathbf{X})$ depends on how the model function $F(\mathbf{z}(t_j;\mathbf{X}))$ is determined. In addition, the underlying form of $F(\mathbf{z}(t_j;\mathbf{X}))$ must be suitably nonlinear to capture the actual nonlinearity of a dynamical system.

3.2 Construction of NARX Model Function

An effective construction of the NARX model function $F(\mathbf{z}(t_j;\mathbf{X}))$ relies on how it is expanded with respect to its argument $\mathbf{z}(t_j;\mathbf{X})$. Denote by $z_i(t_j;\mathbf{X})$ the ith component of $\mathbf{z}(t_j;\mathbf{X})$. There are M such components, depending on the chosen numbers of time lags n_g and n_y.

3.2.1 Dimension-wise Tensor-Product Expansion

For $M\in\mathbb{N}$, let $\{1,\ldots,M\}$ be an index set, so that $u\subseteq\{1,\ldots,M\}$ is a subset, including the emptyset $\emptyset$, with cardinality $0\le|u|\le M$. For a non-empty subset $\emptyset\ne u=\{i_1,\ldots,i_{|u|}\}\subseteq$

$\{1, \ldots, M\}$, denote by

$$\mathbf{z}_u(t_j; \mathbf{X}) = \left\{ z_{i_1}(t_j; \mathbf{X}), \ldots, z_{i_{|u|}}(t_j; \mathbf{X}) \right\}^\intercal$$

a $|u|$-dimensional subvector of $\mathbf{z}(t_j; \mathbf{X})$. Then there exists a finite, hierarchical, convergent expansion of

$$F(\mathbf{z}(t_j; \mathbf{X})) = F_\emptyset(\mathbf{X}) + \sum_{\emptyset \neq u \subseteq \{1,\ldots,M\}} F_u(\mathbf{z}_u(t_j; \mathbf{X})), \tag{4}$$

known as a general dimensional decomposition of a multivariate function. Here, $F_\emptyset(\mathbf{X})$ is a constant, while $F_u(\mathbf{z}_u(t_j; \mathbf{X}))$ is a $|u|$-variate component function describing $|u|$-variate interaction of $\mathbf{z}_u(t_j; \mathbf{X})$ on F.

Given an integer $1 \leq S' \leq M$, an S'-variate approximation, say, $\tilde{F}_{S'}(\mathbf{z}(t_j; \mathbf{X}))$ of $F(\mathbf{z}(t_j; \mathbf{X}))$, is obtained by truncating the right-hand side of Eq. (4), yielding

$$\tilde{F}_{S'}(\mathbf{z}(t_j; \mathbf{X})) = F_\emptyset(\mathbf{X}) + \sum_{\substack{\emptyset \neq u \subseteq \{1,\ldots,M\} \\ 1 \leq |u| \leq S'}} F_u(\mathbf{z}_u(t_j; \mathbf{X})). \tag{5}$$

The approximation is coined "S'-variate," as it retains interactive effects on $F(\mathbf{z}(t_j; \mathbf{X}))$ by at most S' variables. In most applications, a univariate ($S' = 1$) or bivariate ($S' = 2$) truncations suffices for such an approximation.

All $|u|$-variate component functions of $\tilde{F}_{S'}(\mathbf{z}(t_j; \mathbf{X}))$ can be expanded dimension-wise with respect to such a basis, resulting in an S'-variate, m'th-order dimensional decomposition

$$\tilde{y}_{S',m'}(t_j; \mathbf{X}) = F_\emptyset(\mathbf{X}) + \sum_{\substack{\emptyset \neq u \subseteq \{1,\ldots,M\} \\ 1 \leq |u| \leq S'}} \sum_{\substack{\mathbf{j}_u \in \mathbb{N}^{|u|} \\ |u| \leq |\mathbf{j}_u| \leq m'}} R_{u,\mathbf{j}_u}(\mathbf{X}) \Phi_{u,\mathbf{j}_u}(\mathbf{z}_u(t_j; \mathbf{X})), \tag{6}$$

of $\tilde{y}(t_j; \mathbf{X})$, where $R_{u,\mathbf{j}_u}(\mathbf{X}) \in \mathbb{R}$, $\emptyset \neq u \subseteq \{1, \ldots, M\}$, $\mathbf{j}_u \in \mathbb{N}_0^{|u|}$, are the corresponding expansion coefficients and $\Phi_{u,\mathbf{j}_u}(\mathbf{z}_u(t_j; \mathbf{X})$ are the basis functions.

3.2.2 Reduction of No. of NARX Basis Functions

While a dimension-wise construction of NARX model function produces a huge decrease in the number of basis functions already, a further reduction is possible if the coefficients corresponding to some of the basis functions are negligibly small. However, as the NARX expansion coefficients $R_{u,\mathbf{j}_u}(\mathbf{X})$ are functions of input random variables, the magnitude of smallness must be determined in a statistical sense. A four-step algorithm is proposed for basis reduction, as follows.

1. Given the known probability distribution of $\mathbf{X}$, generate $\{\mathbf{x}^{(l)}\}_{l=1,\ldots,L'}$, a set of associated samples of input random variables of size $L' \in \mathbb{N}$.

2. For each input sample $\mathbf{x}^{(l)}$, conduct an appropriate regression analysis by fitting an S'-variate, m'th-order approximation $\tilde{y}_{S',m'}(t_j; \mathbf{x}^{(l)})$ to the original function $y(t_j; \mathbf{x}^{(l)})$, thereby obtaining the corresponding sample of NARX expansion coefficients $R_{u,\mathbf{j}_u}(\mathbf{x}^{(l)})$. More specific details of such regression analysis will be provided in a forthcoming section.

3. Determine the importance of the NARX basis function $\Phi_{u,\mathbf{j}_u}(\mathbf{z}_u(t_j;\mathbf{X}))$ by estimating the mean of the NARX coefficient $R_{u,\mathbf{j}_u}(\mathbf{X})$, referred to as the importance factor, from

$$\bar{\alpha}_{u,\mathbf{j}_u} := \frac{1}{L'}\sum_{l=1}^{L'} \frac{R^2_{u,\mathbf{j}_u}(\mathbf{x}^{(l)})}{\sum\limits_{\substack{\emptyset\neq u\subseteq\{1,\ldots,M\}\\ 1\leq|u|\leq S'}} \sum\limits_{\substack{\mathbf{j}_u\in\mathbb{N}^{|u|}\\ |u|\leq|\mathbf{j}_u|\leq m}} R^2_{u,\mathbf{j}_u}(\mathbf{x}^{(l)})}. \tag{7}$$

According to Eq. (7), the lower the value of $\bar{\alpha}_{u,\mathbf{j}_u}$, the less important the corresponding basis function $\Phi_{u,\mathbf{j}_u}(t_j;\mathbf{z}_u(\mathbf{X}))$.

4. Define a threshold parameter $0 \leq \alpha_0 \leq 1$ for grading the values of $\bar{\alpha}_{u,\mathbf{j}_u}$ from (7). Thereby, remove all basis functions when their importance factors are less than the threshold, that is, from the condition: $\bar{\alpha}_{u,\mathbf{j}_u} \leq \alpha_0$.

It is important to recognize that the magnitude of reduction in the number of NARX basis functions from the proposed algorithm depends on the values of S', m', and α_0 chosen by a user. Furthermore, the reduction also depends on the behavior of original NARX model function $F(\mathbf{z}(t_j;\mathbf{X}))$.

3.3 Estimation of NARX Coefficients

As alluded to earlier, the NARX coefficients are random. Therefore, a UQ method is needed for their statistical characterization. Here, the coefficients are determined in terms of their samples using standard least-square (SLS) regression. Such sample values are required for basis reduction, and will also be needed when PDD is introduced for UQ analysis.

Given an $L'_{S',m'}$ number of basis functions, the NARX approximation can also be expressed by

$$\tilde{y}_{S',m'}(t_j;\mathbf{X}) = \sum_{i=1}^{L'_{S',m'}} R_i(\mathbf{X})\Phi_i(\mathbf{z}(t_j;\mathbf{X})), \tag{8}$$

where $L'_{S',m'}$ is the total number of basis functions determined from the dimensional decomposition parameters S' and m' of the NARX model function and $\mathbf{x}^{(l)}$ is an lth sample, also known as realization or experiment, of $\mathbf{X}$, generated from the known probability distribution of random input. Then, employing SLS regression, the coefficients of the NARX approximation for this particular sample are obtained by minimizing the sum of squared errors,

$$e_{\text{NARX}} := \sum_{j=0}^{J}\left[y(t_j;\mathbf{x}^{(l)}) - \tilde{y}_{S',m'}(t_j;\mathbf{x}^{(l)})\right]^2 = \sum_{j=0}^{J}\left[y(t_j;\mathbf{x}^{(l)}) - \sum_{k=i}^{L'_{S',m'}} R_i(\mathbf{X})\Phi_i(\mathbf{z}(t_j;\mathbf{x}^{(l)}))\right]^2, \tag{9}$$

committed by NARX at all $(J+1)$ time instants t_j, $j = 0,\ldots,J$. The minimization leads to a sample of approximate NARX coefficients

$$\tilde{\mathbf{R}}(\mathbf{x}^{(l)}) := \left\{\tilde{R}_1(\mathbf{x}^{(l)}),\ldots,\tilde{R}_{L'_{S',m'}}(\mathbf{x}^{(l)})\right\}^{\intercal} = (\mathbf{A}_l^{\intercal}\mathbf{A}_l)^{-1}\mathbf{A}_l^{\intercal}\mathbf{b}_l, \tag{10}$$

where

$$\mathbf{A}_l := \begin{bmatrix} \phi_1(\mathbf{z}(t_0;\mathbf{x}^{(l)})) & \cdots & \phi_{L_{S',m'}}(\mathbf{z}(t_0;\mathbf{x}^{(l)})) \\ \vdots & \ddots & \vdots \\ \phi_1(\mathbf{z}(t_J;\mathbf{x}^{(l)})) & \cdots & \phi_{L_{S',m'}}(\mathbf{z}(t_J;\mathbf{x}^{(l)})) \end{bmatrix}, \tag{11}$$

and

$$\mathbf{b}_l := \{y(t_0; \mathbf{x}^{(l)}), \ldots, y(t_J; \mathbf{x}^{(l)}))\}^\intercal. \tag{12}$$

4 PDD

Given an input random vector $\mathbf{X} := \{X_1, \ldots, X_N\}^T : (\Omega, \mathcal{F}) \to (\mathbb{A}^N, \mathcal{B}^N)$ with known PDF $f_{\mathbf{X}}(\mathbf{x})$, let the NARX coefficients $R_i(\mathbf{X}) := R_i(X_1, \ldots, X_N)$, $i = 1, \ldots, L_{\text{NARX}}$ – a real-valued, measurable transformation on $(\Omega, \mathcal{F})$ – describe a stochastic response function of interest. For $\emptyset \neq u = \{i_1, \cdots, i_{|u|}\} \subseteq \{1, \cdots, N\}$, let $\mathbf{X}_u = \{X_{i_1}, \ldots, X_{i_{|u|}}\}^\intercal$, $1 \leq i_1 < \cdots < i_{|u|} \leq N$, be a subvector of $\mathbf{X}$.

A principal objective of UQ analysis is to effectively estimate the relevant probabilistic characteristics of $R_i(\mathbf{X}) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$.

4.1 PDD Approximation

The PDD of a square-integrable random variable $R_i(\mathbf{X}) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ is simply the expansion of $R_i(\mathbf{X})$ with respect to a complete, hierarchically ordered, orthonormal polynomial basis of $L^2(\Omega, \mathcal{F}, \mathbb{P})$. The PDD [4, 5] of $R_i(\mathbf{X})$ is

$$R_i(\mathbf{X}) = R_{i,\emptyset} + \sum_{\emptyset \neq u \subseteq \{1,\ldots,N\}} \sum_{\mathbf{j}_u \in \mathbb{N}^{|u|}} C_{i,u,\mathbf{j}_u} \Psi_{u,\mathbf{j}_u}(\mathbf{X}_u), \tag{13}$$

where,

$$C_{i,u,\mathbf{j}_u} = \int_{\mathbb{A}^N} R_i(\mathbf{x}) \Psi_{u,\mathbf{j}_u}(\mathbf{x}_u) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \tag{14}$$

and $\Psi_{u,\mathbf{j}_u}(\mathbf{X}_u)$, $\mathbf{j}_u = (j_1, \cdots, j_{|u|}) \in \mathbb{N}_0^{|u|}$, a $|u|$-dimensional multi-index, constitutes a measure-consistent multivariate orthonormal polynomial basis of $L^2(\Omega, \mathcal{F}, \mathbb{P})$. The full PDD contains an infinite number of orthonormal polynomials or coefficients. In practice, the number must be finite, meaning that PDD must be truncated. In a practical setting, the function $R_i(\mathbf{X})$ is likely to have an effective dimension much lower than N, meaning that the right side of Eq. (13) can be effectively truncated by a sum of lower-dimensional component functions of PDD, but still preserve all random variables $\mathbf{X}$ of a high-dimensional UQ problem. A straightforward approach adopted in this work entails (1) keeping all polynomials in at most $0 \leq S \leq N$ variables, thereby retaining the degrees of interaction among input variables less than or equal to S and (2) preserving polynomial expansion orders (total) less than or equal to $S \leq m < \infty$. The result is an S-variate, mth-order PDD approximation

$$\tilde{R}_{i,S,m}(\mathbf{X}) = R_{i,\emptyset} + \sum_{\substack{\emptyset \neq u \subseteq \{1,\ldots,N\} \\ 1 \leq |u| \leq S}} \sum_{\substack{\mathbf{j}_u \in \mathbb{N}^{|u|} \\ |u| \leq |\mathbf{j}_u| \leq m}} C_{i,u,\mathbf{j}_u} \Psi_{u,\mathbf{j}_u}(\mathbf{X}_u) \tag{15}$$

of $R_i(\mathbf{X})$, containing

$$L_{S,m} = 1 + \sum_{s=1}^{S} \binom{N}{s} \binom{m}{s} \tag{16}$$

number of expansion coefficients, including $R_{i,\emptyset}$.

4.2 Estimation of PDD Coefficients

The determination of the PDD coefficients $R_{i,\emptyset}$ and $C_{i,u,\mathbf{j}_u}$ involves various N-dimensional integrations. For an arbitrary function $R_i(\mathbf{X})$ and an arbitrary probability distribution of random input $\mathbf{X}$, an exact evaluation of these coefficients from definition alone is impossible. Therefore, a practical alternative to numerical integration, such as regression analysis, is often necessary to estimate these coefficients.

As a result, the S-variate, m-th order PDD approximation can also be written as

$$\tilde{R}_{i,S,m}(\mathbf{X}) = \sum_{k=1}^{L_{S,m}} C_{i,k}\Psi_k(\mathbf{X}), \tag{17}$$

with the expansion coefficients

$$C_{i,k} = \int_{\mathbb{A}^N} R_{i,S,m}(\mathbf{x})\Psi_k(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x},\ k = 1,\ldots,L_{S,m}, \tag{18}$$

and the single index notation of basis functions, $\Psi_k(\mathbf{X})$.

The SLS regression is founded on the optimal approximation quality of the PDD approximation. Given the function $R_i(\mathbf{X})$, select a sample size $L \in \mathbb{N}$ and draw input samples $\mathbf{x}^{(l)}$, $l = 1,\ldots,L$, from the known distribution of random input $\mathbf{X}$. Various sampling methods, namely, standard MCS, quasi MCS, and Latin hypercube sampling, can be used. Corresponding to each input sample, perform full-scale dynamic analysis by solving Eq. (1), thus producing L sets of the dynamic responses. According to SLS, the expansion coefficients of an S-variate, mth-order PDD approximation are estimated by minimizing the empirical analog of the mean-square error

$$e_{\text{PDD}} := \frac{1}{L}\sum_{l=1}^{L}\left[R_i(\mathbf{x}^{(l)}) - \sum_{k=1}^{L_{S,m}} C_{i,k}\Psi_k(\mathbf{x}^{(l)})\right]^2, \tag{19}$$

committed by PDD. Denote by

$$\bar{\mathbf{A}} := \begin{bmatrix} \Psi_1(\mathbf{x}^{(1)}) & \cdots & \Psi_{L_{S,m}}(\mathbf{x}^{(1)}) \\ \vdots & \ddots & \vdots \\ \Psi_1(\mathbf{x}^{(L)}) & \cdots & \Psi_{L_{S,m}}(\mathbf{x}^{(L)}) \end{bmatrix} \tag{20}$$

and

$$\bar{\mathbf{b}} := \left\{R_i(\mathbf{x}^{(1)}),\ldots,R_i(\mathbf{x}^{(L)})\right\}^\intercal \tag{21}$$

an $L \times L_{S,m}$ matrix and an L-dimensional column vector comprising evaluations of the orthonormal polynomial basis functions and output function at the data points, respectively. Then, the estimated coefficients $\tilde{C}_{i,j}$, $j = 1,\ldots,L_{S,m}$, are obtained as

$$\tilde{\mathbf{C}}_i := \left\{\tilde{C}_{i,1},\ldots,\tilde{C}_{i,L_{S,m}}\right\}^\intercal = \left(\bar{\mathbf{A}}^\intercal\bar{\mathbf{A}}\right)^{-1}\bar{\mathbf{A}}^\intercal\bar{\mathbf{b}}. \tag{22}$$

5 Integrated PDD-NARX

The combination of Eqs. (8) and (17), with $R_i(\mathbf{X})$ replaced by $R_{i,S,m}(\mathbf{X})$, results in the desired PDD-NARX approximation:

$$\tilde{y}_{S',m';S,m}(t_j;\mathbf{X}) = \sum_{i=1}^{L'_{S',m'}} \left[\left\{ \sum_{k=1}^{L_{S,m}} C_{i,k}\Psi_k(\mathbf{X}) \right\} \Phi_i(\mathbf{z}(t_j;\mathbf{X})) \right], \tag{23}$$

There are $L'_{S',m'} \times L_{S,m}$ terms in the approximation, depending on the expansion parameters of the NARX model function (S', m') and PDD (S, m) of NARX coefficients.

There is one big advantage of PDD-NARX over PCE-NARX: the number of basis functions of PDD $(L_{S,m})$ grows with N much more slowly than that of PCE $(L_m$, say) if $S \ll N$, raising the potential for solving high-dimensional time-dependent UQ problems.

6 Application

This section highlights the practical application of the PDD-NARX method in tackling a complex vehicle-dynamics problem involving 21 random variables. The focus of the application is dynamic analysis of a Chevrolet C1500 pick-up truck riding over anti-symmetric road bumps, as depicted in Figure 1. The analysis illustrates the capability of the PDD-NARX method to efficiently handle high-dimensional UQ in a real-world engineering scenario, showcasing its potential for addressing complex, large-scale problems.

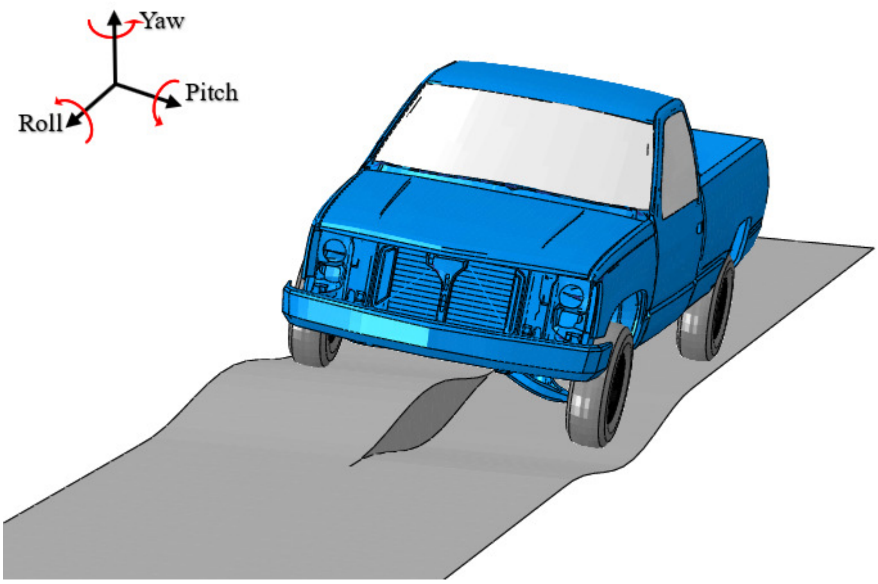


Figure 1: truck riding over anti-symmetric bumps

6.1 Finite Element Analysis

A well-known commercial code for finite element analysis (FEA), named ABAQUS (Version 2024), was utilized to discretize the truck geometry using approximately 55,000 elements.

The dynamic analysis involves the pick-up truck driving over anti-symmetric road bumps. The bumps consist of two random geometric parameters: bump height h_b and bump length l_b. Once the cruise velocity of 8 m/s was achieved, the truck was run over anti-symmetric bumps.

6.2 Material Properties

The materials used in the FEA truck model include steel, plastic, glass, rubber, rubber-metal composite, and foam for representing a wide variety of parts.

Figure 2 provides a color-coded breakdown of the six major different material types utilized in the truck model. For the kth material type ($k = 1, \ldots, 6$), the Young's modulus, Poisson's ratio, and mass density are denoted by E_k, ν_k, and ρ_k, respectively. Additionally, the radial stiffness of the tire is represented by K_R. In total, 19 material parameters were identified as random input in this application. These input material properties are critical for accurately simulating the truck's dynamic behavior and analyzing the response to the road bump interactions.

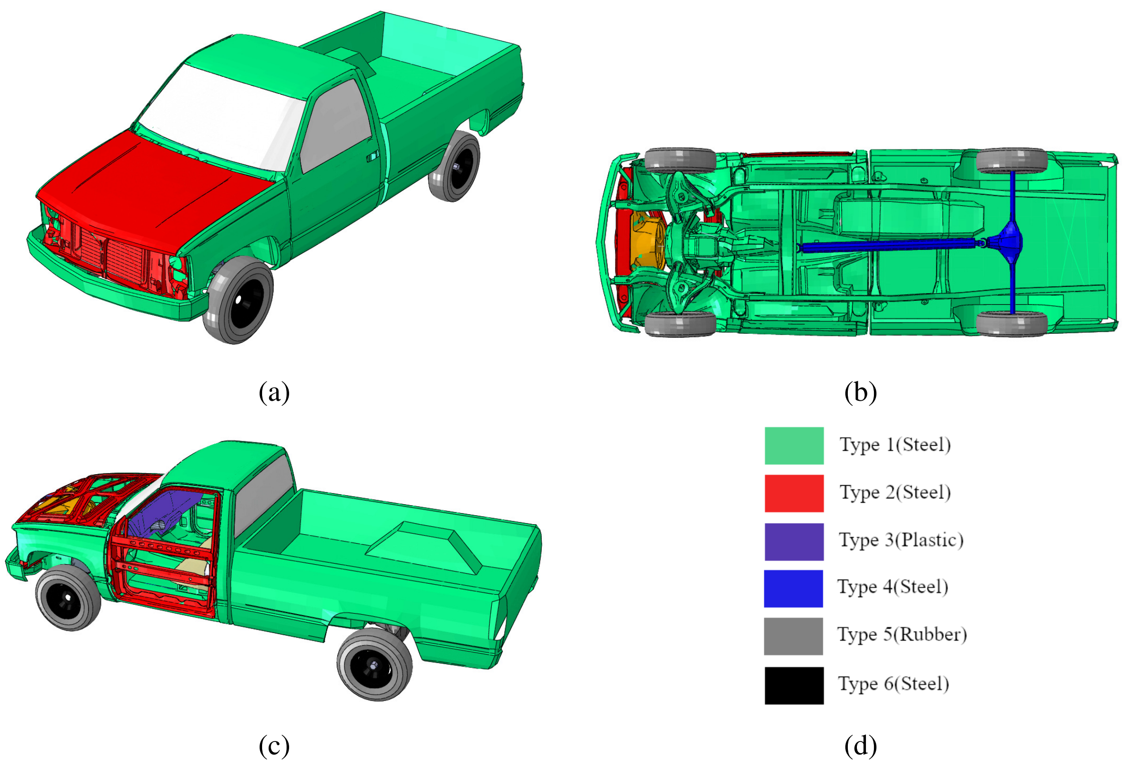


Figure 2: Material distribution in the pick-up truck; (a) isometric view; (b) bottom view; (c) side and inerior view; (d) color legends for six material types.

6.3 Input Random Variables

The input comprises 21 random variables describing two random geometric parameters of the road bumps and 19 random material properties of the truck and tire. The random variables follow independent truncated Gaussian distributions. Their respective means, coefficients of variations, and bounds are provided in Table 1.

Table 1: Mean, coefficient of variation, and bounds of 21 input random variables in the pick-up truck model.

i	Random Variable	Mean (μ_i)	Coefficient of variation (%)	Bounds
1	E_1, MPa	210,000	10	$[0.8\mu_1,1.2\mu_1]$
2	E_2, MPa	210,000	10	$[0.8\mu_2,1.2\mu_2]$
3	E_3, MPa	3400	10	$[0.8\mu_3,1.2\mu_3]$
4	E_4, MPa	210,000	10	$[0.8\mu_4,1.2\mu_4]$
5	E_5, MPa	246,000	10	$[0.8\mu_5,1.2\mu_5]$
6	E_6, MPa	210,000	10	$[0.8\mu_6,1.2\mu_6]$
7	ν_1	0.3	10	$[0.8\mu_7,1.2\mu_7]$
8	ν_2	0.3	10	$[0.8\mu_8,1.2\mu_8]$
9	ν_3	0.3	10	$[0.8\mu_9,1.2\mu_9]$
10	ν_4	0.3	10	$[0.8\mu_{10},1.2\mu_{10}]$
11	ν_5	0.323	10	$[0.8\mu_{11},1.2\mu_{11}]$
12	ν_6	0.3	10	$[0.8\mu_{12},1.2\mu_{12}]$
13	ρ_1, $\mathrm{kg/m^3}$	7890	10	$[0.8\mu_{13},1.2\mu_{13}]$
14	ρ_2, $\mathrm{kg/m^3}$	7890	10	$[0.8\mu_{14},1.2\mu_{14}]$
15	ρ_3, $\mathrm{kg/m^3}$	1100	10	$[0.8\mu_{15},1.2\mu_{15}]$
16	ρ_4, $\mathrm{kg/m^3}$	7890	10	$[0.8\mu_{16},1.2\mu_{16}]$
17	ρ_5, $\mathrm{kg/m^3}$	8060	10	$[0.8\mu_{17},1.2\mu_{17}]$
18	ρ_6, $\mathrm{kg/m^3}$	7890	10	$[0.8\mu_{18},1.2\mu_{18}]$
19	K_R, N/mm	600	10	$[0.8\mu_{19},1.2\mu_{19}]$
20	h_b, mm	200	20	$[0.7\mu_{20},1.3\mu_{20}]$
21	l_b, mm	5000	20	$[0.7\mu_{21},1.3\mu_{21}]$

6.4 Results and Discussion

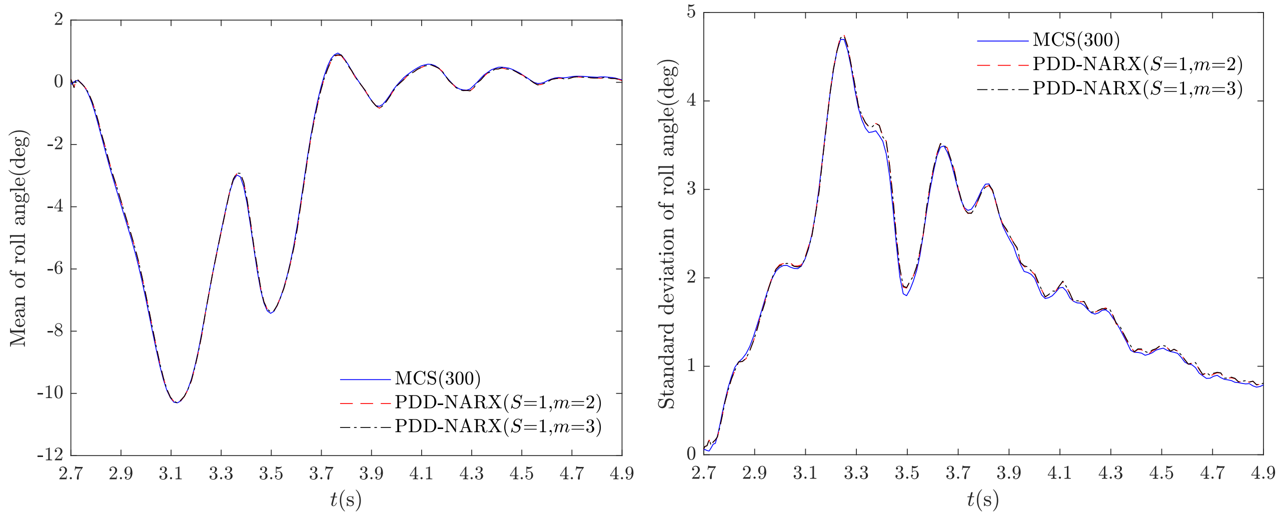
The primary objective of this application is to compute the vehicle body motion angles or rotations, which serve as key indicators of the truck's dynamic behavior. There are three such angles or rotations: roll, pitch, and yaw, as schematically depicted in Figure 1. Therefore, $y(t;\mathbf{X})$ in this work represents any one of these three motion angles at the vehicle's center of gravity (CG).

In this work, 86 or 128 samples of these time histories were used to build the PDD-NARX approximations. Due to the high computational expense of the large-scale simulation, only $L_{MCS} = 300$ samples were used to estimate the second-moment statistics by MCS as the benchmark solution.

6.4.1 Statistical Analysis

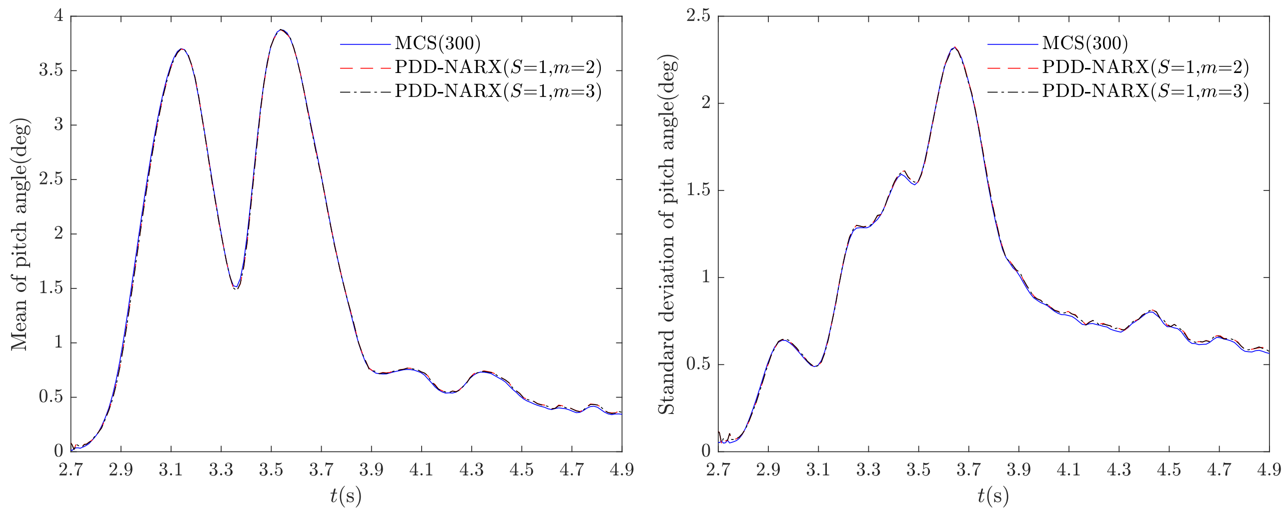
Due to uncertain bump geometry and material properties of the truck, a more realistic evaluation of roll and pitch requires calculating their statistical characteristics. To do so, the NARX model function has to be built first. Using 100 samples of $y(t;\mathbf{X})$, the NARX model function was constructed using the dimension-wise tensor product expansion and basis reduction with the chosen threshold parameter $\alpha_0 = 10^{-8}$.

Given that the yaw angle is small and less oscillatory in this application, the statistical analysis was focused on roll and pitch angles only. Figures 3 and 4 present the second-moment statistics, such as means and variances, of roll and pitch at truck's CG as a function of time, obtained using two variants of the PDD-NARX. The results of both variants of the PDD-NARX are practically conincident, meaning that the second-order approximation is sufficient, at least for this application.



(a) (b)

Figure 3: Second-moment statistics of roll at truck's center of gravity; (a) mean; (b) standard deviation.



(a) (b)

Figure 4: Second-moment statistics of pitch at truck's center of gravity; (a) mean; (b) standard deviation.

7 CONCLUSIONS

The conclusions are as follows:

- PDD-NARX is applicable to both linear and nonlinear dynamical systems, and is distinguished by two key features that are attributed to the NARX and PDD components independently:

    1. A dimension-wise tensor product expansion, coupled with a four-step algorithm for basis reduction, was established for the stochastic adaptation of the NARX model, enabling it to effectively capture the behavior of the dynamical system.

    2. The PDD approximation was exploited to retain low-dimensional interactions among input random variables, which is crucial for efficiently transmitting input uncertainty to the random NARX coefficients.

- The numerical results demonstrates the practicality and scalability of the PDD-NARX method for solving complex, large-scale engineering problems.

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