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ABSTRACT

This paper aims to examine the solvability, uniqueness, and stability of a class of nonlinear implicit fractional Volterra-Fredholm integro-differential equations involving the Caputo-Katugampola fractional derivative. The main objective is to establish rigorous conditions under which such equations admit unique solutions and to analyze their stability behavior under perturbations. Specifically, we apply Banach's fixed-point theorem and a suitable form of the Gronwall inequality to derive sufficient criteria for existence and uniqueness. Furthermore, we investigate the stability of solutions in the sense of Ulam-Hyers and Ulam-Hyers-Rassias, providing a comprehensive understanding of the solution behavior under small deviations. To validate the theoretical results and highlight the applicability of the developed framework, a concrete illustrative example is included. This study contributes to the ongoing development of fractional calculus by deepening the theoretical understanding of fractional integro-differential equations and expanding their potential for modeling complex dynamical systems in various applied sciences.

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1 Introduction

The utilization of fractional calculus has shown remarkable effectiveness in addressing a wide array of scientific and engineering problems. Notable advancements concerning fractional derivatives can be found in [1–9]. Progress continues to be made in both fractional partial differential equations and ordinary differential equations. The study of fractional differential equations has been extensively developed in recent literature. In [10], the authors investigate nonlinear fractional implicit differential equations, establishing existence and uniqueness results through fixed-point theorems and fractional calculus techniques.

Building on fixed-point theory, study [11] proves a Krasnoselskii-Schaefer type theorem that combines contraction and compactness conditions for operator equations. The theoretical foundations for

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such analysis can be found in [12], which presents a comprehensive treatment of functional analysis in applied mathematics.

Recent applications to fractional order equations include the work in [13] on semilinear functional differential equations and [14] on nonlinear boundary value problems, both employing fixed-point methods. A broader survey of these techniques for fractional boundary value problems is provided in [15].

The concept of stability in functional equations originated from a lecture by Ulam at the University of Wisconsin in 1940, where he questioned the conditions under which an approximate additive function is near an exact additive one, as detailed in [16]. Hyers provided an initial answer in 1941 within the framework of Banach spaces [17], leading to what is now referred to as Ulam-Hyers stability. Later, in 1978, Rassias expanded this idea by introducing perturbations involving additional parameters [11].

In research on functional equations, stability is typically examined by substituting the original equation with an inequality, accounting for deviations or perturbations. Thus, the primary concern of stability analysis lies in quantifying the gap between the solutions of the perturbed problem and those of the ideal equation. Numerous comprehensive texts discuss Ulam-Hyers and Ulam-Hyers-Rassias stability across a broad spectrum of functional equations [18–20].

Among the latest innovations in fractional calculus is the Caputo-Katugampola fractional derivative, which generalizes and refines concepts from both the Hadamard and Riemann-Liouville derivatives. Introduced by Katugampola, this operator merges the frameworks of the Caputo and Caputo-Hadamard derivatives [21]. Beyond the established Caputo derivative, the Caputo-Katugampola derivative introduces a sophisticated integral-based structure [10,11]. Recent years have witnessed a surge of interest in this operator, with significant findings reported in [22–27].

Recent advances in fractional integro differential equations have expanded to include various types of integro-differential problems. In [28], the authors establish novel results on positive solutions for nonlinear Caputo-Hadamard fractional Volterra integro-differential equations, providing important existence criteria. The study [29] extends these investigations to neutral fractional Volterra-Fredholm integro-differential equations, proving existence, uniqueness, and stability results under general conditions. Further developments appear in [30], where new results are presented for Caputo fractional Volterra-Fredholm integro-differential equations with nonlocal conditions, expanding the theoretical framework for such problems.

This work explores the existence, uniqueness, and stability (Ulam-Hyers and Ulam-Hyers-Rassias) of solutions to fractional differential equations incorporating the Caputo-Katugampola derivative operator. For further related studies, see [31]. The fundamental equation under consideration is:

$$\mathfrak{D}_{\mathfrak{b}^+}^{\mathfrak{w},\kappa}\wp(\mathfrak{s}) = \mathfrak{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathfrak{D}_{\mathfrak{b}^+}^{\mathfrak{w},\kappa}\wp(\mathfrak{s})), \mathfrak{s} \in \psi = [\mathfrak{b}, \xi],$$
$$\wp(\mathfrak{b}) = \wp_0.$$

Fractional integro-differential equations are crucial in modeling various applications, including viscoelasticity, population dynamics, and control systems. The Caputo-Katugampola fractional derivative enhances classical methods, allowing more accurate modeling of memory effects. This work establishes solvability conditions for implicit fractional Volterra-Fredholm equations and ensures their stability through rigorous mathematical analysis.

The purpose of this study is to introduce a new analytical framework for addressing nonlinear fractional Volterra-Fredholm integro-differential equations by utilizing the Caputo-Katugampola fractional derivative. This approach is designed to generalize and unify various types of fractional derivatives, allowing for greater flexibility in modeling real-world phenomena. We also aim to establish existence and uniqueness results under appropriate conditions and support the theoretical findings with illustrative examples and numerical simulations.

This paper is based on the previous study that was referred to in the introduction, which is an extension of fractional differential equations to fractional integro differential equations under the influence of Caputo-Katugampola Operator, see [13].

In this paper, we investigate the existence, uniqueness and Ulam-Hyers and Ulam-Hyers-Rassias stability of implicit fractional Volterra-Fredholm integro-differential equations within the framework of the Caputo-Katugampola fractional derivative.

$$\mathfrak{D}^{\mathfrak{w},\kappa}_{\mathfrak{b}^+}\wp(\mathfrak{s}) = \mathfrak{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathfrak{D}^{\mathfrak{w},\kappa}_{\mathfrak{b}^+}\wp(\mathfrak{s}), (\mathcal{K}\wp)(\mathfrak{s}), (\mathcal{H}\wp)(\mathfrak{s})), \mathfrak{s} \in \psi = [\mathfrak{b}, \xi], \tag{1}$$

$$\wp(\mathfrak{b}) = \wp_0, \tag{2}$$

where $\wp \in \mathcal{C}_{\varepsilon}[\mathfrak{b}, \xi]$, denotes the weighted space of continuous functions satisfying the specified fractional smoothness conditions $\mathfrak{s} \in [\mathfrak{b}, \xi]$, $\mathfrak{w} \in (0, 1]$, and $\kappa > 0$, $\wp_0$ is a constant. The function $\mathfrak{F} : \psi \times \mathbb{R}^{\mathfrak{d}} \times \mathbb{R}^{\mathfrak{d}} \times \mathbb{R}^{\mathfrak{d}} \times \mathbb{R}^{\mathfrak{d}} \longrightarrow \mathbb{R}^{\mathfrak{d}}$ is a nonlinear, continuously differentiable vector-valued function, and $\mathfrak{D}^{\mathfrak{w},\kappa}_{\mathfrak{b}^+}$ denotes the Caputo–Katugampola fractional derivative and

$$(\mathcal{K}\wp)(\mathfrak{s}) = \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s}, \sigma)\wp(\sigma)d\sigma, \quad (\mathcal{H}\wp)(\mathfrak{s}) = \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s}, \sigma)\wp(\sigma)d\sigma$$

where $\mathfrak{K}, \mathfrak{h} : \Delta \times \psi \longrightarrow \mathbb{R}$, $\Delta = \{(\mathfrak{s}, \sigma) : [\mathfrak{b} \le \sigma \le \mathfrak{s} \le \xi\}$

Reason for Choosing Caputo-Katugampola

- This derivative generalizes both the Caputo and Hadamard fractional derivatives, offering a flexible framework with an additional parameter (ρ) that allows tuning the operator for specific applications (e.g., modeling non-local phenomena with variable memory effects).
- It is particularly useful for problems where the kernel's singularity or weighting needs adaptation, such as in viscoelasticity, anomalous diffusion, or systems with power-law memory.

Differences from Other Derivatives

- Caputo: The Caputo-Katugampola derivative incorporates a weight function $t^{\rho-1}$ in its kernel, enabling it to interpolate between Caputo ($\rho \to 1$) and Hadamard-type ($\rho \to 0^+$) operators.
- Riemann-Liouville: Unlike Riemann-Liouville, the Caputo-Katugampola formulation avoids initial conditions with fractional derivatives, making it physically interpretable for real-world problems.
- Atangana-Baleanu: While Atangana-Baleanu uses Mittag-Leffler kernels for non-singularity, Caputo-Katugampola retains a power-law kernel but with ρ-controlled flexibility, often simplifying analytical solutions.

The remaining content is structured as follows: Section 2 presents with the fundamental ideas and theorems that will underpin the results. The existence and uniqueness of solutions (1) and (2) of the system under appropriate assumptions are demonstrated in Section 3. In Section 4, we will also discuss the Ulam-Hyers-Rassias stability of our problem. Example is subsequently detailed in Section 5.

2 Auxiliary Results

This section introduces FDE theory words, concepts, and Lemmas. These will be used to draw primary conclusions later in the study. Additionally, we reviewed the Banach fixed point Theorem. Our discussion is primarily concerned with the function space that follows, see [32].

Consider the interval $[\mathfrak{b},\xi]\subset\mathbb{R}$, where $0<\mathfrak{b}<\xi<\infty$, and let $\mathcal{C}([\mathfrak{b},\xi],\ \mathbb{R}^{\mathfrak{d}})$ constitute the domain of all continuous functions with vector values $\wp:[\mathfrak{b},\xi]\longrightarrow\mathbb{R}^{\mathfrak{d}}$. Then, $\mathcal{C}([\mathfrak{b},\xi],\mathbb{R}^{\mathfrak{d}})$ is a Banach space when equipped with the

$$\|\wp\|_{\mathcal{C}([\mathfrak{b},\xi],\mathbb{R}^{\mathfrak{d}})}=\sup\{|\wp(\mathfrak{s})|:\mathfrak{s}\in[\mathfrak{b},\xi]\},$$

where $|\cdot|$ represents the vector norm in $\mathbb{R}^{\mathfrak{d}}$.

Suppose $\mathcal{C}^{m}([\mathfrak{b},\xi],\mathbb{R}^{\mathfrak{d}})$ be the vector-valued function's space. $\wp$ including a continuous derivative of $m-$order, where $\wp:[\mathfrak{b},\xi]\longrightarrow\mathbb{R}^{\mathfrak{d}}$.

The following works by considering the weighted space of continuous functions:

$$\mathcal{C}_{\varepsilon}(\psi,\mathbb{R}^{\mathfrak{d}})=\left\{\wp:[\mathfrak{b},\xi]\to\mathbb{R}^{\mathfrak{d}}:\left(\frac{\mathfrak{s}^{\kappa}-\mathfrak{b}^{\kappa}}{\kappa}\right)^{\varepsilon}\wp(\mathfrak{s})\in\mathcal{C}\big(\psi,\mathbb{R}^{\mathfrak{d}}\big)\right\},\varepsilon\in(0,1].$$

Definition 1: *([27]) Let $\wp\in\mathcal{C}(\psi,\mathbb{R}^{\mathfrak{d}}),\mathfrak{s}\in(\mathfrak{b},\xi]$ and parameter $\kappa>0$. Let $\nu=\lceil\mathfrak{w}\rceil+1$. Then, the left and right Caputo–Katugampola fractional derivatives of order $\mathfrak{w}>0$ are defined:*

$$^{\mathcal{C}}\mathfrak{D}^{\mathfrak{w},\kappa}_{\mathfrak{b}^+}\wp(\mathfrak{s})=\frac{\kappa^{\mathfrak{w}-\nu+1}}{\Gamma(\nu-\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\frac{\mathfrak{s}^{(\kappa-1)(1-\nu)}}{(\mathfrak{s}^{\kappa}-\varpi^{\kappa})^{\nu-\mathfrak{w}+1}}\wp^{(\nu)}(\varpi)\,d\varpi,$$
$$^{\mathcal{C}}\mathfrak{D}^{\mathfrak{w},\kappa}_{\mathfrak{b}^-}\wp(\mathfrak{s})=\frac{(-1)^{\nu}\kappa^{\mathfrak{w}-\nu+1}}{\Gamma(\nu-\mathfrak{w})}\int_{\mathfrak{s}}^{\mathfrak{b}}\frac{\mathfrak{s}^{(\kappa-1)(1-\nu)}}{(\mathfrak{s}^{\kappa}-\varpi^{\kappa})^{\nu-\mathfrak{w}+1}}\wp^{(\nu)}(\varpi)\,d\varpi.$$

Definition 2: *([27]) Let $\wp\in\mathcal{C}(\psi,\mathbb{R}^{\mathfrak{d}}),\mathfrak{s}\in(\mathfrak{b},\xi],\mathfrak{w}\in(0,1)$ and parameter $\kappa>0$. The Katugampola fractional integralis given by:*

$$\mathfrak{I}^{\mathfrak{w},\kappa}_{\mathfrak{b}^+}\wp(\mathfrak{s})=\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\frac{\mathfrak{s}^{\kappa-1}}{(\mathfrak{s}^{\kappa}-\varpi^{\kappa})^{1-\mathfrak{w}}}\wp(\varpi)\,d\varpi,$$
$$\mathfrak{I}^{\mathfrak{w},\kappa}_{\mathfrak{b}^-}\wp(\mathfrak{s})=\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{s}}^{\mathfrak{b}}\frac{\mathfrak{s}^{\kappa-1}}{(\mathfrak{s}^{\kappa}-\varpi^{\kappa})^{1-\mathfrak{w}}}\wp(\varpi)\,d\varpi.$$

Definition 3: *([26]) For $\wp\in\mathcal{C}(\psi,\mathbb{R}^{\mathfrak{d}})$, where $\psi=(\mathfrak{b},\xi)$ and $\mathfrak{w}\in(0,1)$, the Riemann-Liouville generalized fractional integral is defined as:*

$$\mathfrak{I}^{\mathfrak{w},\kappa}_{\mathfrak{b}}\wp(\mathfrak{s})=\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\frac{\varpi^{\kappa-1}}{(\mathfrak{s}^{\kappa}-\varpi^{\kappa})^{1-\mathfrak{w}}}\wp(\varpi)\,d\varpi.$$

Definition 4: *([4]) Let $\wp\in\mathcal{C}^{\nu}(\psi,\mathbb{R}^{\mathfrak{d}})$, where $\psi=(\mathfrak{b},\xi]$, and let $\nu=\lceil\mathfrak{w}\rceil$ be the least integer greater than or equal to $\mathfrak{w}$. The generalized Katugampola fractional operator is defined by:*

$$\left(\mathfrak{I}^{\mathfrak{w},\kappa}_{\mathfrak{b}^+}\mathfrak{D}^{\mathfrak{w},\kappa}_{\mathfrak{b}^+}\wp\right)(\mathfrak{s})=\wp(\mathfrak{s})-\sum_{m=0}^{\nu-1}\frac{\wp^{(\kappa,m)}(\mathfrak{b})}{m!}\left(\frac{\mathfrak{s}^{\kappa}-\mathfrak{b}^{\kappa}}{\kappa}\right)^{m},\ \mathfrak{s}\in(\mathfrak{b},\xi],$$

where the generalized higher-order derivative at $\mathfrak{b}$ *is given by:*

$$\wp^{(\kappa,m)}(\mathfrak{b}) = \left(\left[\mathfrak{s}^{1-\kappa}\left(\frac{d}{d\mathfrak{s}}\right)\right]^m \wp(\mathfrak{s})\right)_{\mathfrak{s}=\mathfrak{b}}.$$

Here:

- $\wp^{(\kappa,m)}(\mathfrak{b})$ *represents the eneralized Caputo-Katugampola derivatives evaluated at* $\mathfrak{b}$.
- m *is the smallest integer satisfying* $m \geq \mathfrak{w}$.
- *The summation term accounts for the correction needed to ensure consistency with the classical function recovery property.*

Lemma 1: *[4](Gronwall's Inequality") On the interval* $[\mathfrak{b}, \xi]$*, there are two non-negative functions* $\wp(\mathfrak{s})$ *and* $\hbar(\mathfrak{s})$ *that are integrable. Additionally, there is a continuous, non-negative, and non-decreasing function on the same interval, denoted as* $\rho(\mathfrak{s})$.

$$\wp(\mathfrak{s}) \leq \hbar(\mathfrak{s}) + \rho(\mathfrak{s})\kappa^{1-\mathfrak{w}} \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{s}^{\kappa-1}(\mathfrak{s}^\kappa - \varpi^\kappa)^{\mathfrak{w}-1}\wp(\varpi)\, d\varpi, \forall \mathfrak{s} \in (\mathfrak{b}, \xi],$$

then

$$\wp(\mathfrak{s}) \leq \hbar(\mathfrak{s}) + \int_{\mathfrak{b}}^{\mathfrak{s}} \sum_{m=0}^{\infty} \frac{\kappa^{1-\ell\mathfrak{w}}(\rho(\mathfrak{s})\Gamma(\ell))^\ell}{\Gamma(\ell\mathfrak{w})} \mathfrak{s}^{\kappa-1}(\mathfrak{s}^\kappa - \varpi^\kappa)^{\ell\mathfrak{w}-1}\wp(\varpi)\, d\varpi, \forall \mathfrak{s} \in (\mathfrak{b}, \xi].$$

Moreover, if the function $\hbar(\mathfrak{s})$ *is nondecreasing, then*

$$\wp(\mathfrak{s}) \leq \hbar(\mathfrak{s}) E_{\mathfrak{w},1}\left(\rho(\mathfrak{s})\Gamma(m)\left(\frac{\mathfrak{s}^\kappa - \mathfrak{b}^\kappa}{\kappa}\right)^{\mathfrak{w}}\right), \forall \mathfrak{s} \in (\mathfrak{b}, \xi],$$

where the Mittag-Leffler function is defined as:

$$E_{\mathfrak{w},1}(\mathfrak{s}) = \sum_{m=0}^{\infty} \frac{\mathfrak{s}^m}{\Gamma(m\mathfrak{w}+1)}.$$

Theorem 1: *(Banach's fixed point Theorem)*

Let (Ω) *be a non-empty complete metric space, and let* $\Phi : \Omega \to \Omega$ *be a contraction mapping. Then,* Φ *has a unique fixed point* $\mu \in \Omega$ *satisfying* $\Phi(\mu) = \mu$.

Lemma 2: *Let* $\mathfrak{p} : \psi \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ *is a continuous function. Then the linear problem*

$$\mathfrak{D}^{\mathfrak{w},\kappa}_{\mathfrak{b}^+}\wp(\mathfrak{s}) = \mathfrak{p}_{\wp}(\mathfrak{s}), \mathfrak{s} \in \psi = [\mathfrak{b}, \xi],$$

$$\wp(\mathfrak{b}) = \wp_0,$$

where

$$\mathfrak{p}(\mathfrak{s}) = \mathfrak{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathfrak{D}^{\mathfrak{w},\kappa}_{\mathfrak{b}^+}\wp(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma), \mathfrak{s} \in \psi = [\mathfrak{b}, \xi].$$

The function $\wp(\mathfrak{s})$ *resolves (1) if and only if it satisfies the integral equation provided by*

$$\wp(\mathfrak{s}) = \wp_0 + \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})} \int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1} \mathfrak{p}_{\wp}(\varpi)d\varpi. \tag{3}$$

Proof: To solve the differential equation, we convert it into an integral equation. The Caputo–Katugampola fractional integral of order $\mathfrak{w}$ is given by:

$$\mathfrak{I}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp(\mathfrak{s}) = \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}(\mathfrak{s}^{\kappa}-\varpi^{\kappa})^{\mathfrak{w}-1}\varpi^{\kappa-1}\wp(\varpi)d\varpi.$$

Applying the fractional integral operator $\mathfrak{I}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}$ to both sides of the differential equation, we obtain:

$$\wp(\mathfrak{s}) = \wp_0 + \mathfrak{I}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\mathfrak{p}_{\wp}(\mathfrak{s}).$$

Substitute the expression for $\mathfrak{p}(\mathfrak{s})$ into the integral equation:

$$\wp(\mathfrak{s}) = \wp_0 + \mathfrak{I}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\mathfrak{F}\left(\mathfrak{s},\wp(\mathfrak{s}),\mathfrak{D}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma\right).$$

The integral equation is:

$$\wp(\mathfrak{s}) = \wp_0 + \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}(\mathfrak{s}^{\kappa}-\varpi^{\kappa})^{\mathfrak{w}-1}\varpi^{\kappa-1}\mathfrak{F}\Bigg(\varpi,\wp(\varpi),\mathfrak{D}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp(\varpi),\int_{\mathfrak{b}}^{\varpi}\mathfrak{K}(\varpi,\tau)\wp(\tau)d\tau,$$
$$\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\varpi,\tau)\wp(\tau)d\tau\Bigg)d\varpi.$$

Therefore

$$\wp(\mathfrak{s}) = \wp_0 + \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^{\kappa}-\varpi^{\kappa}\right)^{\mathfrak{w}-1}\ \mathfrak{p}_{\wp}(\varpi)d\varpi.$$

This is the desired integral equation that satisfies the original Caputo–Katugampola fractional differential equation. □

3 Main Results

To demonstrate the main conclusions, we need the following presumptions:

(Λ_1) $\mathfrak{F}:\psi\times\mathbb{R}^{\mathfrak{d}}\times\mathbb{R}^{\mathfrak{d}}\times\mathbb{R}^{\mathfrak{d}}\times\mathbb{R}^{\mathfrak{d}}\longrightarrow\mathbb{R}^{\mathfrak{d}}$

(Λ_2) There exist non-negative constants $\delta_1 > 0, 0 < \delta_2 < 1, \delta_3, \delta_4 > 0, \forall\mathfrak{s}\in\psi$ such that the function $\mathfrak{F}$ satisfies

$$\|\mathfrak{F}(\mathfrak{s},\wp_1,\wp_2,\wp_3,\wp_4)-\mathfrak{F}(\mathfrak{s},\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3,\mathfrak{v}_4)\| \le \delta_1\|\wp_1-\mathfrak{v}_1\|+\delta_2\|\wp_2-\mathfrak{v}_2\|$$
$$+\delta_3\|\wp_3-\mathfrak{v}_3\|+\delta_4\|\wp_4-\mathfrak{v}_4\|$$

(Λ_3) There exist two constants $\mathcal{K}^*$ and $\mathcal{H}^*$ such that

$$\mathcal{K}^* = \left|\sup_{\mathfrak{s}\in\psi}\int_{\mathfrak{b}}^{\mathfrak{s}}\left|\mathfrak{K}(\mathfrak{s},\sigma)d\sigma\right|<\infty,\quad \mathcal{H}^* = \left|\sup_{\mathfrak{s}\in\psi}\int_{\mathfrak{b}}^{\xi}\left|\mathfrak{h}(\mathfrak{s},\sigma)d\sigma\right|<\infty$$

(Λ_4) There exists $\mu,\nu,\zeta,\eta,\tau\in\mathfrak{C}(\psi,\mathbb{R})$,with

$$\mu^*=\sup_{\mathfrak{s}\in\psi}\mu(\mathfrak{s})<1,\quad \nu^*=\sup_{\mathfrak{s}\in\psi}\nu(\mathfrak{s})<1,\quad \zeta^*=\sup_{\mathfrak{s}\in\psi}\zeta(\mathfrak{s})<1,$$
$$\eta^*=\sup_{\mathfrak{s}\in\psi}\eta(\mathfrak{s})<1,\quad \tau^*=\sup_{\mathfrak{s}\in\psi}\tau(\mathfrak{s})<1,$$ **then**

$$\mathfrak{F}(\mathfrak{s},\wp_1,\wp_2,\wp_3,\wp_4) \leq \mu(\mathfrak{s}) + \nu(\mathfrak{s})|\wp_1| + \zeta(\mathfrak{s})|\wp_2| + \eta(\mathfrak{s})|\wp_3| + \tau(\mathfrak{s})|\wp_4|$$

Theorem 2: *Let the assumptions (Λ_1), (Λ_2) and (Λ_3) be true. If*

$$\Xi := \frac{\left[\delta_1 + \delta_3\mathcal{K}^* + \delta_4\mathcal{H}^*\right]\kappa^{-\mathfrak{w}}(\mathfrak{s}^\kappa - \varpi^\kappa)^{\mathfrak{w}}}{(1-\delta_2)\Gamma(\mathfrak{w}+1)} < 1, \tag{4}$$

then the system (1) and (2) has a unique solution on $\mathfrak{s} \in \mathfrak{C}(\psi,\mathbb{R})$.

Proof: The operator defined $\Omega : \mathfrak{C}(\psi,\mathbb{R}) \longrightarrow \mathfrak{C}(\psi,\mathbb{R})$

$$\Omega\wp(\mathfrak{s}) = \wp_0 + \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1} \quad \mathfrak{p}_\wp(\varpi)d\varpi \tag{5}$$

for any $\wp,\wp_0 \in \mathfrak{C}(\psi,\mathbb{R})$, we have,

$$\begin{aligned}|(\Omega\wp)(\mathfrak{s}) - (\Omega\wp_0)(\mathfrak{s})| &= \left|\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1} \quad \mathfrak{p}(\varpi)d\varpi \right.\\ &\quad \left. - \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1} \quad \mathfrak{p}_0(\varpi)d\varpi\right|\\ &\leq \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1}|\mathfrak{p}(\varpi) - \mathfrak{p}_0(\varpi)|d\varpi, \end{aligned} \tag{6}$$

where $\mathfrak{p},\mathfrak{p}_0 \in \mathfrak{C}(\psi,\mathbb{R})$ be such that

$$\mathfrak{p}(\mathfrak{s}) = \mathfrak{F}\left(\mathfrak{s},\wp(\mathfrak{s}),\mathfrak{p}(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma\right)$$

and

$$\mathfrak{p}_0(\mathfrak{s}) = \mathfrak{F}\left(\mathfrak{s},\wp_0(\mathfrak{s}),\mathfrak{p}_0(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp_0(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp_0(\sigma)d\sigma\right).$$

By using (Λ_3), we find

$$\left|\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma - \int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp_0(\sigma)d\sigma\right| \leq \left|\sup_{\mathfrak{s}\in\psi}\int_{\mathfrak{b}}^{\mathfrak{s}}\left|\mathfrak{K}(\mathfrak{s},\sigma)d\sigma\right|\right|\wp - \wp_0|$$

$$\leq \mathcal{K}^*\|\wp - \wp_0\|$$

and

$$\left|\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma - \int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp_0(\sigma)d\sigma\right| \leq \left|\sup_{\mathfrak{s}\in\psi}\int_{\mathfrak{b}}^{\xi}\left|\mathfrak{h}(\mathfrak{s},\sigma)d\sigma\right|\right|\wp - \wp_0|$$

$$\leq \mathcal{H}^*\|\wp - \wp_0\|.$$

From (Λ_2) we get

$$\begin{aligned}\Big|\mathfrak{p}(\mathfrak{s})-\mathfrak{p}_0(\mathfrak{s})\Big| &= \Big|\mathfrak{F}\Big((\mathfrak{s},\wp(\mathfrak{s}),\mathfrak{p}((\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma\Big)\\ &\quad-\mathfrak{F}\Big((\mathfrak{s},\wp_0(\mathfrak{s}),\mathfrak{p}_0(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp_0(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp_0(\sigma)d\sigma\Big)\Big|\\ &\le \delta_1\big|\wp(\mathfrak{s})-\wp_0(\mathfrak{s})\big|+\delta_2\big|\mathfrak{p}(\mathfrak{s})-\mathfrak{p}_0(\mathfrak{s})\big|+\big[\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\big]|\wp-\wp_0|.\end{aligned}$$

Thus

$$\Big|\mathfrak{p}(\mathfrak{s})-\mathfrak{p}_0(\mathfrak{s})\Big| \le \frac{\delta_1+\big[\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\big]}{1-\delta_2}|\wp-\wp_0|. \tag{7}$$

Replacing (7) in (6), we obtain

$$\begin{aligned}|(\Omega\wp)(\mathfrak{s})-(\Omega\wp_0)(\mathfrak{s})| &\le \frac{\delta_1+\big[\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\big]}{1-\delta_2}\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\big(\mathfrak{s}^{\kappa}-\varpi^{\kappa}\big)^{\mathfrak{w}-1}|\wp(\varpi)-\wp_0(\varpi)|d\varpi\\ &\le \frac{\delta_1+\big[\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\big]}{(1-\delta_2)\Gamma(\mathfrak{w})}|\wp(\mathfrak{s})-\wp_0(\mathfrak{s})|\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\frac{\mathfrak{s}^{\kappa}-\varpi^{\kappa}}{\kappa}\right)^{\mathfrak{w}-1}d\varpi\\ &\le \frac{\big[\delta_1+\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\big]\kappa^{-\mathfrak{w}}(\mathfrak{s}^{\kappa}-\varpi^{\kappa})^{\mathfrak{w}}}{(1-\delta_2)\Gamma(\mathfrak{w}+1)}|\wp(\mathfrak{s})-\wp_0(\mathfrak{s})|.\end{aligned}$$

Hence,

$$\|\Omega\wp-\Omega\wp_0\| \le \Xi\|\wp-\wp_0\|.$$

From Eqs. (1) and (2), Ω is a contraction. As a consequence of Banach's fixed point Theorem, we get that Ω has a unique fixed point which is a unique solution of the problem (1) and (2). □

After that, we apply Schauder's fixed point Theorem to the second outcome and examine it.

Theorem 3: *Assumptions* $(\Lambda_1)-(\Lambda_4)$ *is true, then the problem (1) and (2) has at least one valid solution.*

Proof: Consider $\Upsilon_{\mathfrak{q}}=\{\wp:\wp\in\mathfrak{C}(\psi,\mathbb{R}^{\mathfrak{d}}),|\wp(\mathfrak{s})|\le\mathfrak{q}\}$. We finalize the proof through a series of steps.

Step I: Ω is a continuous mapping. Suppose $\wp_m,\wp\in\Upsilon_{\Omega}$ a sequence $\{\wp_m\}$ converging to $\{\wp_m\}\to\wp$ *as* $m\to\infty$ in $\mathfrak{C}(\psi,\mathbb{R})$. For each $\mathfrak{s}\in\psi$, we can observe that:

$$\begin{aligned}|(\Omega\wp)(\mathfrak{s})-(\Omega\wp_m)(\mathfrak{s})| &= \Big|\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\big(\mathfrak{s}^{\kappa}-\varpi^{\kappa}\big)^{\mathfrak{w}-1}\ \mathfrak{p}(\varpi)d\varpi\\ &\quad-\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\big(\mathfrak{s}^{\kappa}-\varpi^{\kappa}\big)^{\mathfrak{w}-1}\ \mathfrak{p}_0(\varpi)d\varpi\Big|\\ &\le \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\big(\mathfrak{s}^{\kappa}-\varpi^{\kappa}\big)^{\mathfrak{w}-1}|\mathfrak{p}(\varpi)-\mathfrak{p}_m(\varpi)|d\varpi,\end{aligned} \tag{8}$$

where $\mathfrak{p}, \mathfrak{p}_m \in \mathfrak{C}(\psi, \mathbb{R})$ satisfies the functional equation

$$\mathfrak{p}(\mathfrak{s}) = \mathfrak{F}\left(\mathfrak{s}, \wp(\mathfrak{s}), \mathfrak{p}(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma\right)$$

and

$$\mathfrak{p}_m(\mathfrak{s}) = \mathfrak{F}\left(\mathfrak{s}, \wp_m(\mathfrak{s}), \mathfrak{p}_m(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s},\sigma)\wp_m(\sigma)d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s},\sigma)\wp_m(\sigma)d\sigma\right).$$

By using $(\Lambda_2) - (\Lambda_3)$, we find

$$\begin{aligned}\left|\mathfrak{p}(\mathfrak{s}) - \mathfrak{p}_m(\mathfrak{s})\right| &= \left|\mathfrak{F}\left((\mathfrak{s}, \wp(\mathfrak{s}), \mathfrak{p}((\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma\right)\right.\\ &\quad \left. - \mathfrak{F}\left((\mathfrak{s}, \wp_0(\mathfrak{s}), \mathfrak{p}_m(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s},\sigma)\wp_m(\sigma)d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s},\sigma)\wp_m(\sigma)d\sigma\right)\right|\\ &\le \delta_1\left|\wp(\mathfrak{s}) - \wp_0(\mathfrak{s})\right| + \delta_2\left|\mathfrak{p}(\mathfrak{s}) - \mathfrak{p}_0(\mathfrak{s})\right| + \left[\delta_3\mathcal{K}^* + \delta_4\mathcal{H}^*\right]|\wp - \wp_0|.\end{aligned}$$

Thus

$$\left|\mathfrak{p}(\mathfrak{s}) - \mathfrak{p}_m(\mathfrak{s})\right| \le \frac{\delta_1 + \left[\delta_3\mathcal{K}^* + \delta_4\mathcal{H}^*\right]}{1-\delta_2}|\wp - \wp_m|. \tag{9}$$

Thus, $\wp_m \to \wp, \mathfrak{p}_m \to \mathfrak{p}$ as $m \to \infty$, and $(\Omega\wp_m)(\mathfrak{s}) \to (\Omega\wp)(\mathfrak{s})$. On the other hand Ω is continuous.

Step II: $\Omega(\Upsilon_{\mathfrak{q}}) \subset \Upsilon_{\mathfrak{q}}$

This step proves that the operator Ω maps the closed ball $\Upsilon_{\mathfrak{q}}$ (of radius $\mathfrak{q}$) into itself, which is essential for applying fixed-point theorems.

Definition of Ω and Goal

The operator Ω is defined as:

$$(\Omega\wp)(\mathfrak{s}) = \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}(\mathfrak{s}^{\kappa} - \varpi^{\kappa})^{\mathfrak{w}-1}\mathfrak{p}_{\wp}(\varpi)\, d\varpi, \tag{10}$$

where $\mathfrak{p}_{\wp}(\varpi)$ is a nonlinear term depending on $\wp$.

Goal: Show that if $\wp \in \Upsilon_{\mathfrak{q}}$ (i.e., $\|\wp\| \le \mathfrak{q}$), then $\Omega\wp \in \Upsilon_{\mathfrak{q}}$.

Bound on the Nonlinear Term $\mathfrak{p}_{\wp}(\mathfrak{s})$

The term $\mathfrak{p}_{\wp}(\mathfrak{s})$ is given by:

$$\mathfrak{p}_{\wp}(\mathfrak{s}) = \mathfrak{F}\left(\mathfrak{s}, \wp(\mathfrak{s}), \mathfrak{p}_{\wp}(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)\, d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)\, d\sigma\right).$$

Under Assumption (Λ_4), $\mathfrak{F}$ satisfies:

$$|\mathfrak{p}_{\wp}(\mathfrak{s})| \le \mu(\mathfrak{s}) + \nu(\mathfrak{s})|\wp(\mathfrak{s})| + \zeta(\mathfrak{s})|\mathfrak{p}_{\wp}(\mathfrak{s})| + \mathcal{K}^*\eta(\mathfrak{s})|\wp| + \mathcal{H}^*\tau(\mathfrak{s})\|\wp\|.$$

Solving for $|\mathfrak{p}_{\wp}(\mathfrak{s})|$, we obtain:

$$|\mathfrak{p}_{\wp}(\mathfrak{s})| \le \frac{\mu(\mathfrak{s}) + \nu(\mathfrak{s})|\wp(\mathfrak{s})| + \mathcal{K}^*\eta(\mathfrak{s})|\wp| + \mathcal{H}^*\tau(\mathfrak{s})\|\wp\|}{1-\zeta(\mathfrak{s})}.$$

Taking suprema over $\mathfrak{s}$ and defining:

$$\mu^* = \|\mu\|_\infty, \quad \nu^* = \|\nu\|_\infty, \quad \zeta^* = \|\zeta\|_\infty, \quad \eta^* = \|\eta\|_\infty, \quad \tau^* = \|\tau\|_\infty,$$

we get the uniform bound:

$$|\mathfrak{p}_\wp(\mathfrak{s})| \le \frac{\mu^* + (\nu^* + \mathcal{K}^*\eta^* + \mathcal{H}^*\tau^*)\mathfrak{q}}{1-\zeta^*}. \tag{11}$$

Estimating $|(\Omega\wp)(\mathfrak{s})|$

Substitute (11) into (10):

$$|(\Omega\wp)(\mathfrak{s})| \le \frac{\mu^* + (\nu^* + \mathcal{K}^*\eta^* + \mathcal{H}^*\tau^*)\mathfrak{q}}{1-\zeta^*} \cdot \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})} \int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}(\mathfrak{s}^\kappa - \varpi^\kappa)^{\mathfrak{w}-1}\, d\varpi.$$

The integral simplifies via substitution $u = \varpi^\kappa$:

$$\int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}(\mathfrak{s}^\kappa - \varpi^\kappa)^{\mathfrak{w}-1}\, d\varpi = \kappa^{-1}(\mathfrak{s}^\kappa - \mathfrak{b}^\kappa)^{\mathfrak{w}} \frac{\Gamma(\mathfrak{w})\Gamma(1)}{\Gamma(\mathfrak{w}+1)}.$$

Thus:

$$|(\Omega\wp)(\mathfrak{s})| \le \left[\frac{\mu^* + (\nu^* + \mathcal{K}^*\eta^* + \mathcal{H}^*\tau^*)\mathfrak{q}}{1-\zeta^*}\right] \frac{\kappa^{-\mathfrak{w}}(\mathfrak{s}^\kappa - \mathfrak{b}^\kappa)^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)}.$$

Conclusion: Ω Preserves $\Upsilon_{\mathfrak{q}}$

For $\Omega\wp \in \Upsilon_{\mathfrak{q}}$, we require:

$$\left[\frac{\mu^* + (\nu^* + \mathcal{K}^*\eta^* + \mathcal{H}^*\tau^*)\mathfrak{q}}{1-\zeta^*}\right] \frac{\kappa^{-\mathfrak{w}}(\xi^\kappa - \mathfrak{b}^\kappa)^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)} \le \mathfrak{q}.$$

This inequality determines the admissible radius $\mathfrak{q}$ of $\Upsilon_{\mathfrak{q}}$.

Step III: We demonstrate that the expression $\Omega(\Upsilon_{\mathfrak{q}})$ is equicontinuous.

It is clear from **step II** that $\Omega(\Upsilon_{\mathfrak{q}}) \subset \Upsilon_{\mathfrak{q}}$ is bounded. $\mathfrak{s}_1, \mathfrak{s}_2 \in (1, \psi]$, $\mathfrak{s}_1 < \mathfrak{s}_2$ and let $\mathfrak{s} \in \Upsilon_{\mathfrak{q}}$. Then

$$\begin{aligned}
&|(\Omega\wp)(\mathfrak{s}_2) - (\Omega\wp)(\mathfrak{s}_1)| \\
&= \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\left| \int_{\mathfrak{b}}^{\mathfrak{s}_1} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}_2{}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1} \mathfrak{p}_\wp(\varpi)d\varpi - \int_{\mathfrak{b}}^{\mathfrak{s}_2} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}_1{}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1} \mathfrak{p}_\wp(\varpi)d\varpi \right| \\
&= \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\left| \int_{\mathfrak{b}}^{\mathfrak{s}_1} \left[\left(\mathfrak{s}_2{}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1} - \left(\mathfrak{s}_1{}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1}\right] \varpi^{\mathfrak{w}-1}\mathfrak{p}_\wp(\varpi)d\varpi \right. \\
&+ \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\left| \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}_2{}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1} \mathfrak{p}_\wp(\varpi)d\varpi \right| \\
&\le \frac{|\mathfrak{p}_\wp(\varpi)|\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\left| \int_{\mathfrak{b}}^{\mathfrak{s}_1} \left[\left(\mathfrak{s}_2{}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1} - \left(\mathfrak{s}_1{}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1}\right] \varpi^{\mathfrak{w}-1}d\varpi \right|
\end{aligned}$$

$$+\frac{|\mathfrak{p}_{\wp}(\varpi)|\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\left|\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}_2{}^{\kappa}-\varpi^{\kappa}\right)^{\mathfrak{w}-1}\ d\varpi\right|$$

$$\leq\frac{\mu^*+(\nu^*+\mathcal{K}^*\eta^*+\mathcal{H}^*\tau^*)\mathfrak{q}}{1-\zeta^*}\cdot\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\left|\int_{\mathfrak{b}}^{\mathfrak{s}_1}\left[\left(\mathfrak{s}_2{}^{\kappa}-\varpi^{\kappa}\right)^{\mathfrak{w}-1}-\left(\mathfrak{s}_1{}^{\kappa}-\varpi^{\kappa}\right)^{\mathfrak{w}-1}\right]\ \varpi^{\mathfrak{w}-1}d\varpi\right|$$

$$+\frac{\mu^*+(\nu^*+\mathcal{K}^*\eta^*+\mathcal{H}^*\tau^*)\mathfrak{q}}{1-\zeta^*}\cdot\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\left|\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}_2{}^{\kappa}-\varpi^{\kappa}\right)^{\mathfrak{w}-1}\ d\varpi\right|$$

$$\leq\frac{\mu^*+(\nu^*+\mathcal{K}^*\eta^*+\mathcal{H}^*\tau^*)\mathfrak{q}}{(1-\zeta^*)\Gamma(\mathfrak{w}+1)}\left[\kappa^{-\mathfrak{w}}\left((\mathfrak{s}_1{}^{\kappa}-\mathfrak{b}^{\kappa})^{\mathfrak{w}}-(\mathfrak{s}_2{}^{\kappa}-\mathfrak{b}^{\kappa})^{\mathfrak{w}}\right)\right]$$

$$\longrightarrow 0\quad \text{as}\,\mathfrak{s}_1\longrightarrow\mathfrak{s}_2.$$

The Arzela-Ascoli Theorem shows that Ω is relatively compact in both scenarios, and Schauder's fixed point Theorem states that Ω has a fixed point. Then, Ω is a solution of system (1) and (2). □

4 Ulam–Hyers–Rassias Stability

Definition 5: *For any* $\mathfrak{q}>0$, *let the function* $\wp^*\in\mathfrak{C}^1([\mathfrak{b},\xi],\mathbb{R}^{\mathfrak{d}})$, *fulfills the inequality*

$$\left|\mathfrak{D}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp^*(\mathfrak{s})-\mathfrak{F}\left(\mathfrak{s},\wp^*(\mathfrak{s}),\mathfrak{D}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp^*(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma\right)\right|\leq\mathfrak{q}(\mathfrak{s}),\mathfrak{s}\in[\mathfrak{b},\xi]\tag{12}$$

If there are real numbers $\mathcal{M}_{\mathfrak{F}}>0$, $\Xi_{\mathfrak{F}}>0$ *and a solution* $\wp$ *of Eq. (1), such that*

$$\left|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})\right|\leq\mathcal{M}_{\mathfrak{F}\mathfrak{q}}E_{\mathfrak{w},1}\left(\Xi_{\mathfrak{F}}\left(\frac{\mathfrak{s}^{\kappa}-\mathfrak{b}^{\kappa}}{\kappa}\right)^{\mathfrak{w}}\right),\mathfrak{s}\in[\mathfrak{b},\xi]$$

then Eq. (1) is Ulam–Hyers stable.

Definition 6: *For all* $\mathfrak{q}>0$, *suppose that the function* $\aleph$ *belongs to* $\mathfrak{C}([\mathfrak{b},\xi],\mathbb{R}^{\mathfrak{d}})$ *and the function* $\wp^*\in\mathfrak{C}^1([\mathfrak{b},\xi],\mathbb{R}^{\mathfrak{d}})$ *satisfies the inequality*

$$\left|\mathfrak{D}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp^*(\mathfrak{s})-\mathfrak{F}\left(\mathfrak{s},\wp^*(\mathfrak{s}),\mathfrak{D}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp^*(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma\right)\right|$$
$$\leq\mathfrak{q}(\mathfrak{s})\aleph(\mathfrak{s}),\mathfrak{s}\in[\mathfrak{b},\xi]\tag{13}$$

Assuming that actual integers Eq. (1) has a solution $\wp$ *and* $\mathcal{M}_{\mathfrak{F}}>0$, *so that*

$$\left|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})\right|\leq\mathcal{M}_{\mathfrak{F}\mathfrak{q}}E_{\mathfrak{w},1}\left(\Xi_{\mathfrak{F}}\left(\frac{\mathfrak{s}^{\kappa}-\mathfrak{b}^{\kappa}}{\kappa}\right)^{\mathfrak{w}}\right)\aleph(\mathfrak{s}),\mathfrak{s}\in[\mathfrak{b},\xi]$$

the Ulam-Hyers-Rassias stability of Eq. (1) with regard to $\aleph$.

Definition 7: *For any* $\aleph\in\mathfrak{C}([\mathfrak{b},\xi],\mathbb{R}^{\mathfrak{d}})$, *and the function* $\wp^*\in\mathfrak{C}^1([\mathfrak{b},\xi],\mathbb{R}^{\mathfrak{d}})$, *fulfills the inequality*

$$\left|\mathfrak{D}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp^*(\mathfrak{s})-\mathfrak{F}\left(\mathfrak{s},\wp^*(\mathfrak{s}),\mathfrak{D}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp^*(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma\right)\right|\leq\aleph(\mathfrak{s}),\mathfrak{s}\in[\mathfrak{b},\xi]\tag{14}$$

If there exist real numbers $\mathcal{M}_{\mathfrak{F}} > 0, \Xi_{\mathfrak{F}} > 0$ and a solution $\wp$ of Eq. (1), such that

$$\left|\wp^*(\mathfrak{s}) - \wp(\mathfrak{s})\right| \leq \mathcal{M}_{\mathfrak{F}\mathfrak{q}} E_{\mathfrak{w},1}\left(\Xi_{\mathfrak{F}}\left(\frac{\mathfrak{s}^\kappa - \mathfrak{b}^\kappa}{\kappa}\right)^{\mathfrak{w}}\right)\aleph(\mathfrak{s}), \forall \mathfrak{s} \in [\mathfrak{b}, \xi]$$

the Ulam-Hyers-Rassias stability of Eq. (1) with regard to $\aleph$.

Definition 8: *Let function $\wp^* \in \mathfrak{C}^1([\mathfrak{b}, \xi], \mathbb{R}^{\mathfrak{d}})$. satisfy the inequality (12), and if there exists a function $\Lambda_{\mathfrak{F}} \in \mathfrak{C}(\mathbb{R}^+, \mathbb{R}^+)$, $\Lambda_{\mathfrak{F},\wp^*} > 0$ and a solution $\wp^*$ of Eq. (1), such that*

$$\left|\wp^*(\mathfrak{s}) - \wp(\mathfrak{s})\right| \leq \Lambda_{\mathfrak{F},\wp^*}(\mathfrak{q}) E_{\mathfrak{w},1}\left(\Xi_{\mathfrak{F}}\left(\frac{\mathfrak{s}^\kappa - \mathfrak{b}^\kappa}{\kappa}\right)^{\mathfrak{w}}\right), \mathfrak{s} \in [\mathfrak{b}, \xi], \quad \Xi_{\mathfrak{F}} \geq 0$$

then Eq. (1) is generalized Ulam–Hyers stable.

Theorem 4: *Assuming that the function (1) is associated with (Λ_1), (Λ_2), and (Λ_3), then Eq. (1) is Ulam–Hyers stable.*

Proof: Suppose that $\wp^* \in \mathfrak{C}^1([\mathfrak{b}, \xi]$, is a solution to the in Eq. (12), that is,

$$\left|\mathfrak{D}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp^*(\mathfrak{s}) - \mathfrak{F}\left(\mathfrak{s}, \wp^*(\mathfrak{s}), \mathfrak{D}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp^*(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s}, \delta)\wp^*(\sigma)d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s}, \sigma)\wp^*(\sigma)d\sigma\right)\right| \leq \mathfrak{q}(\mathfrak{s}), \forall \mathfrak{s} \in [\mathfrak{b}, \xi] \quad (15)$$

Let us designate by $\wp \in \mathfrak{C}^1([\mathfrak{b}, \xi])$ the unique solution of Eq. (1).

$$\mathfrak{D}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp(\mathfrak{s}) = \mathfrak{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathfrak{D}_{\mathfrak{b}+}^{\mathfrak{w},\kappa}\wp(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s}, \sigma)\wp(\sigma)d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s}, \sigma)\wp(\sigma)d\sigma), \mathfrak{s} \in \psi = [\mathfrak{b}, \xi],$$

$$\wp(\mathfrak{b}) = \wp_0^*$$

Using Lemma 2, we get

$$\wp(\mathfrak{s}) = \wp_0 + \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1} \quad \mathfrak{p}_\wp(\varpi)d\varpi$$

where $\mathfrak{p}_\wp \in \mathfrak{C}^1([\mathfrak{b}, \xi])$ satisfies the functional equation.

$$\mathfrak{p}_\wp(\mathfrak{s}) = \mathfrak{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathfrak{p}_\wp(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s}, \sigma)\wp(\sigma)d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s}, \sigma)\wp(\sigma)d\sigma)$$

Then, by integration of (12),

$$\left|\wp^*(\mathfrak{s}) - \wp_0^* - \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}-1} \quad \mathfrak{p}_\wp^*(\varpi)d\varpi\right| \leq \frac{\mathfrak{q}\kappa^{-\mathfrak{w}}\left(\mathfrak{s}^\kappa - \varpi^\kappa\right)^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)} \quad (16)$$

where $\mathfrak{p}_\wp^* \in \mathfrak{C}^1([\mathfrak{b}, \xi])$ satisfies the functional equation.

$$\mathfrak{p}_\wp^*(\mathfrak{s}) = \mathfrak{F}(\mathfrak{s}, \wp^*(\mathfrak{s}), \mathfrak{p}_\wp^*(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s}, \sigma)\wp^*(\sigma)d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s}, \sigma)\wp^*(\sigma)d\sigma), \quad \forall \mathfrak{s} \in [\mathfrak{b}, \xi]$$

We have

$$\begin{aligned}
\left|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})\right| &= \left|\wp^*(\mathfrak{s})-\wp_0^*-\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa-\varpi^\kappa\right)^{\mathfrak{w}-1}\ \mathfrak{p}_\wp(\varpi)d\varpi\right| \\
&= \left|\wp^*(\mathfrak{s})-\wp_0^*-\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa-\varpi^\kappa\right)^{\mathfrak{w}-1}\ \mathfrak{p}^*_\wp(\varpi)d\varpi\right. \\
&\left.+\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa-\varpi^\kappa\right)^{\mathfrak{w}-1}\ (\mathfrak{p}^*_\wp(\varpi)-\mathfrak{p}_\wp(\varpi))d\varpi\right| \\
&\leq \left|\wp^*(\mathfrak{s})-\wp_0^*-\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa-\varpi^\kappa\right)^{\mathfrak{w}-1}\ \mathfrak{p}^*_\wp(\varpi)d\varpi\right| \\
&+\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa-\varpi^\kappa\right)^{\mathfrak{w}-1}\ |\mathfrak{p}^*_\wp(\varpi)-\mathfrak{p}_\wp(\varpi))|d\varpi
\end{aligned} \tag{17}$$

where

$$\mathfrak{p}^*_\wp(\mathfrak{s})=\mathfrak{F}(\mathfrak{s},\wp^*(\mathfrak{s}),\mathfrak{p}^*_\wp(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma),\quad \forall\mathfrak{s}\in[\mathfrak{b},\xi])$$

and

$$\mathfrak{p}_\wp(\mathfrak{s})=\mathfrak{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathfrak{p}_\wp(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma),\quad \forall\mathfrak{s}\in[\mathfrak{b},\xi])$$

From $(\Lambda_2)-(\Lambda_3)$ we get

$$\begin{aligned}
\left|\mathfrak{p}^*_\wp(\mathfrak{s})-\mathfrak{p}_\wp(\mathfrak{s})\right| &= \left|\mathfrak{F}\Big((\mathfrak{s},\wp^*(\mathfrak{s}),\mathfrak{p}^*_\wp((\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma\Big)\right. \\
&\left.-\mathfrak{F}\Big((\mathfrak{s},\wp(\mathfrak{s}),\mathfrak{p}_\wp(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma\Big)\right| \\
&\leq \delta_1\left|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})\right|+\delta_2\left|\mathfrak{p}^*_\wp(\mathfrak{s})-\mathfrak{p}_\wp(\mathfrak{s})\right|+\left[\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\right]|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})|
\end{aligned}$$

Thus

$$\left|\mathfrak{p}^*_\wp(\mathfrak{s})-\mathfrak{p}_\wp(\mathfrak{s})\right| \leq \frac{\delta_1+\left[\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\right]}{1-\delta_2}|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})| \tag{18}$$

Thus, by (16)–(18) we obtain

$$\begin{aligned}
\left|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})\right| &\leq \frac{\mathfrak{q}\kappa^{-\mathfrak{w}}\left(\mathfrak{s}^\kappa-\varpi^\kappa\right)^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)} \\
&+\frac{\left[\delta_1+\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\right]\kappa^{1-\mathfrak{w}}}{(1-\delta_2)\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^\kappa-\varpi^\kappa\right)^{\mathfrak{w}-1}|\wp^*(\varpi)-\wp(\varpi)|d\varpi
\end{aligned}$$

By using Gronwall inequality, we obtain

$$\left|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})\right| \leq \frac{\mathfrak{q}\kappa^{-\mathfrak{w}}\left(\mathfrak{s}^\kappa-\varpi^\kappa\right)^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)}E_{\mathfrak{w},1}\left(\frac{\left[\delta_1+\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\right]}{(1-\delta_2)}\left(\frac{\mathfrak{s}^\kappa-\varpi^\kappa}{\kappa}\right)^{\mathfrak{w}}\right),\quad \mathfrak{s}\in[\mathfrak{b},\xi]$$

where

$$\mathcal{M}_{\mathfrak{F}} = \frac{\kappa^{-\mathfrak{w}}\left(\mathfrak{s}^{\kappa} - \varpi^{\kappa}\right)^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)}, \quad \Xi_{\mathfrak{F}} = \frac{\left[\delta_1 + \delta_3\mathcal{K}^* + \delta_4\mathcal{H}^*\right]}{(1-\delta_2)}$$

Therefore, the function (1) is stable according to Ulam-Hyers. The proof is now complete. So far, by putting $\Lambda_{\mathfrak{F}}(\mathfrak{q}) = \frac{\mathfrak{q}\kappa^{-\mathfrak{w}}\left(\mathfrak{s}^{\kappa} - \varpi^{\kappa}\right)^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)}$, $\Lambda_{\mathfrak{F}}(0) = 0$, $\Xi_{\mathfrak{F}} = \frac{\left[\delta_1 + \delta_3\mathcal{K}^* + \delta_4\mathcal{H}^*\right]}{(1-\delta_2)}$ allows for the generalization of Eq. (1) is Ulam-Hyers Stable. □

Theorem 5: *Assuming that the function (1) satisfies conditions (Λ_1), (Λ_2), and (Λ_3) simultaneously, the function $\aleph \in \mathfrak{C}([\mathfrak{b}, \xi], \mathbb{R}^{\mathfrak{d}})$ is increasing, and there exists $\mu_{\aleph} > 0$ such that for all $\mathfrak{s} \in [\mathfrak{b}, \xi]$, we have*

$$\mathfrak{I}^{\mathfrak{w},\kappa}_{\mathfrak{b}+}\aleph(\mathfrak{s}) \leq \mu_{\aleph}\aleph(\mathfrak{s}).$$

Then, Eq. (1) is Ulam–Hyers–Rassias stable with respect to $\aleph$.

Proof: Let function $\wp^* \in \mathfrak{C}^1([\mathfrak{b}, \xi]$, be a solution of the in Eq. (13), i.e.,

$$\left|\mathfrak{D}^{\mathfrak{w},\kappa}_{\mathfrak{b}+}\wp^*(\mathfrak{s}) - \mathfrak{F}\left(\mathfrak{s}, \wp^*(\mathfrak{s}), \mathfrak{D}^{\mathfrak{w},\kappa}_{\mathfrak{b}+}\wp^*(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s},\delta)\wp^*(\delta)d\delta, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s},\delta)\wp^*(\delta)d\delta\right)\right| \leq \mathfrak{q}\aleph(\mathfrak{s}), \mathfrak{s} \in [\mathfrak{b}, \xi] \tag{19}$$

Let us denote by $\wp \in \mathfrak{C}^1([\mathfrak{b}, \xi]$, the unique solution of Eq. (1)

$$\mathfrak{D}^{\mathfrak{w},\kappa}_{\mathfrak{b}+}\wp(\mathfrak{s}) = \mathfrak{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathfrak{D}^{\mathfrak{w},\kappa}_{\mathfrak{b}+}\wp(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma), \mathfrak{s} \in \psi = [\mathfrak{b}, \xi],$$

$$\wp_0 = \wp_0^*$$

By using Lemma 2, we have

$$\wp(\mathfrak{s}) = \wp_0 + \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^{\kappa} - \varpi^{\kappa}\right)^{\mathfrak{w}-1} \mathfrak{p}_{\wp}(\varpi)d\varpi$$

where $\mathfrak{p}_{\wp} \in \mathfrak{C}^1([\mathfrak{b}, \xi])$ satisfies the functional equation.

$$\mathfrak{p}_{\wp}(\mathfrak{s}) = \mathfrak{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathfrak{p}_{\wp}(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma)$$

Then, by integration of (19) and by Theorem 4,

$$\begin{aligned}\left|\wp^*(\mathfrak{s}) - \wp_0^* - \frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^{\kappa} - \varpi^{\kappa}\right)^{\mathfrak{w}-1} \mathfrak{p}^*_{\wp}(\varpi)d\varpi\right| &\leq \frac{\mathfrak{q}\kappa^{-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}} \varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^{\kappa} - \varpi^{\kappa}\right)^{\mathfrak{w}-1}\aleph(\varpi)d\varpi \\ &\leq \mathfrak{q}\mu_{\aleph}\aleph(\mathfrak{s})\end{aligned} \tag{20}$$

where $\mathfrak{p}^*_{\wp} \in \mathfrak{C}^1([\mathfrak{b}, \xi])$ satisfies the functional equation.

$$\mathfrak{p}^*_{\wp}(\mathfrak{s}) = \mathfrak{F}(\mathfrak{s}, \wp^*(\mathfrak{s}), \mathfrak{p}^*_{\wp}(\mathfrak{s}), \int_{\mathfrak{b}}^{\mathfrak{s}} \mathfrak{K}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma, \int_{\mathfrak{b}}^{\xi} \mathfrak{h}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma), \quad \forall \mathfrak{s} \in [\mathfrak{b}, \xi]$$

We have

$$\left|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})\right| = \left|\wp^*(\mathfrak{s})-\wp_0^*-\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^{\kappa}-\varpi^{\kappa}\right)^{\mathfrak{w}-1}\ \mathfrak{p}_{\wp}(\varpi)d\varpi\right|$$

$$= \left|\wp^*(\mathfrak{s})-\wp_0^*-\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^{\kappa}-\varpi^{\kappa}\right)^{\mathfrak{w}-1}\ \mathfrak{p}^*_{\wp}(\varpi)d\varpi\right.$$

$$\left.+\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^{\kappa}-\varpi^{\kappa}\right)^{\mathfrak{w}-1}\ (\mathfrak{p}^*_{\wp}(\varpi)-\mathfrak{p}_{\wp}(\varpi))d\varpi\right|$$

$$\leq \left|\wp^*(\mathfrak{s})-\wp_0^*-\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^{\kappa}-\varpi^{\kappa}\right)^{\mathfrak{w}-1}\ \mathfrak{p}^*_{\wp}(\varpi)d\varpi\right|$$

$$+\frac{\kappa^{1-\mathfrak{w}}}{\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^{\kappa}-\varpi^{\kappa}\right)^{\mathfrak{w}-1}\ |\mathfrak{p}^*_{\wp}(\varpi)-\mathfrak{p}_{\wp}(\varpi))|d\varpi \tag{21}$$

where

$$\mathfrak{p}^*_{\wp}(\mathfrak{s})=\mathfrak{F}(\mathfrak{s},\wp^*(\mathfrak{s}),\mathfrak{p}^*_{\wp}(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma),\quad \forall\mathfrak{s}\in[\mathfrak{b},\xi])$$

and

$$\mathfrak{p}_{\wp}(\mathfrak{s})=\mathfrak{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathfrak{p}_{\wp}(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma),\quad \forall\mathfrak{s}\in[\mathfrak{b},\xi])$$

From $(\Lambda_2)-(\Lambda_3)$ we get

$$\left|\mathfrak{p}^*_{\wp}(\mathfrak{s})-\mathfrak{p}_{\wp}(\mathfrak{s})\right| = \left|\mathfrak{F}\left((\mathfrak{s},\wp^*(\mathfrak{s}),\mathfrak{p}^*_{\wp}((\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp^*(\sigma)d\sigma\right)\right.$$

$$\left.-\mathfrak{F}\left((\mathfrak{s},\wp(\mathfrak{s}),\mathfrak{p}_{\wp}(\mathfrak{s}),\int_{\mathfrak{b}}^{\mathfrak{s}}\mathfrak{K}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma,\int_{\mathfrak{b}}^{\xi}\mathfrak{h}(\mathfrak{s},\sigma)\wp(\sigma)d\sigma\right)\right|$$

$$\leq \delta_1\left|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})\right|+\delta_2\left|\mathfrak{p}^*_{\wp}(\mathfrak{s})-\mathfrak{p}_{\wp}(\mathfrak{s})\right|+\left[\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\right]|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})|$$

Thus

$$\left|\mathfrak{p}^*_{\wp}(\mathfrak{s})-\mathfrak{p}_{\wp}(\mathfrak{s})\right| \leq \frac{\delta_1+\left[\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\right]}{1-\delta_2}|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})| \tag{22}$$

Thus, by (20)–(22) we obtain

$$\left|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})\right| \leq \mathfrak{q}\mu_{\aleph}\aleph(\mathfrak{s})$$

$$+\frac{\left[\delta_1+\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\right]\kappa^{1-\mathfrak{w}}}{(1-\delta_2)\Gamma(\mathfrak{w})}\int_{\mathfrak{b}}^{\mathfrak{s}}\varpi^{\mathfrak{w}-1}\left(\mathfrak{s}^{\kappa}-\varpi^{\kappa}\right)^{\mathfrak{w}-1}|\wp^*(\varpi)-\wp(\varpi)|d\varpi$$

By using Gronwall inequality, we obtain

$$\left|\wp^*(\mathfrak{s})-\wp(\mathfrak{s})\right| \leq \mathfrak{q}\mu_{\aleph}\aleph(\mathfrak{s})E_{\mathfrak{w},1}\left(\frac{\left[\delta_1+\delta_3\mathcal{K}^*+\delta_4\mathcal{H}^*\right]}{(1-\delta_2)}\left(\frac{\mathfrak{s}^{\kappa}-\varpi^{\kappa}}{\kappa}\right)^{\mathfrak{w}}\right),\quad \mathfrak{s}\in[\mathfrak{b},\xi]$$

where

$$\mathcal{M}_{\mathfrak{F}} = \mu_{\aleph}, \quad \Xi_{\mathfrak{F}} = \frac{\left[\delta_1 + \delta_3\mathcal{K}^* + \delta_4\mathcal{H}^*\right]}{(1-\delta_2)}$$

Therefore, based on the Ulam–Hyers–Rassias stability criterion, the function (1) is stable. This completes the proof.

By setting $\mathfrak{q} = 1$, $\mathcal{M}_{\mathfrak{F}} = \mu_{\aleph}$, and

$$\Xi_{\mathfrak{F}} = \frac{\left[\delta_1 + \delta_3\mathcal{K}^* + \delta_4\mathcal{H}^*\right]}{(1-\delta_2)},$$

- The choice $\mathfrak{q} = 1$ reduces the generalized stability to the classical Ulam–Hyers–Rassias case
- The constant $\mathcal{M}_{\mathfrak{F}} = \mu_{\aleph}$ links the stability bound to the system's fundamental parameters
- The expression for $\Xi_{\mathfrak{F}}$ incorporates all relevant system parameters:
  - δ_1 represents the base stability constant
  - δ_2 accounts for the contractive nature of the operator
  - $\delta_3\mathcal{K}^*$ and $\delta_4\mathcal{H}^*$ capture the influence of the Volterra and Fredholm terms, respectively.

It is shown that the generalized Ulam–Hyers–Rassias stability is established by Eq. (1). □

This parameterization shows that the generalized Ulam–Hyers–Rassias stability is fully characterized by Eq. (1).

5 Application

To illustrate the practical applicability of our theoretical results, we consider a specific example of a nonlinear implicit fractional Volterra–Fredholm integro-differential equation. This example demonstrates how the conditions derived in the previous sections ensure the existence, uniqueness, and stability of solutions for such problems.

5.1 Example

Consider the following nonlinear implicit fractional Volterra–Fredholm integro-differential equation:

$$\begin{aligned} {}^{C}\mathfrak{D}_0^{\frac{2}{3},4}\wp(\mathfrak{s}) = {} & \frac{4 + |\wp(\mathfrak{s})| + |{}^{C}\mathfrak{D}_0^{\frac{2}{3},4}\wp(\mathfrak{s})|}{4e^{\mathfrak{s}+3}\left(4 + |\wp(\mathfrak{s})| + |{}^{C}\mathfrak{D}_0^{\frac{2}{3},4}\wp(\mathfrak{s})|\right)} + \frac{\mathfrak{b}}{4}\int_0^{\mathfrak{s}} |(\mathfrak{s}-\sigma+1)|\frac{|\sin(\wp(\sigma))|}{3+|\wp(\sigma)|}d\sigma \\ & + \int_0^{\mathfrak{b}} e^{-(\mathfrak{s}+2\sigma)}\frac{|\tan(\wp(\sigma))|}{2+|\wp(\sigma)|}d\sigma, \quad \forall \mathfrak{s} \in [0,1] \end{aligned} \tag{23}$$

subject to the initial condition:

$$\wp(0) = 1, \tag{24}$$

where $\mathfrak{w} = \frac{2}{3}$, $\kappa = 4$, and $\mathfrak{b} = 1$. Define the nonlinear operator $\mathfrak{F}$ as:

$$\mathfrak{F}(\mathfrak{s}, \wp_1, \wp_2, \wp_3, \wp_4) = \frac{4 + |\wp_1| + |\wp_2|}{4e^{\mathfrak{s}+3}\left(4 + |\wp_1| + |\wp_2|\right)} + \frac{|\sin(\wp_3)|}{5 + |\wp_3|} + \frac{|\tan(\wp_4)|}{4 + |\wp_3|}.$$

It is evident that $\mathfrak{F}$ is continuous, and for any $\wp_1, \wp_2, \wp_3, \wp_4, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4 \in \mathbb{R}^{\mathfrak{d}}$ and $\mathfrak{s} \in [0,1]$, the following Lipschitz condition holds:

$$\|\mathfrak{F}(\mathfrak{s}, \wp_1, \wp_2, \wp_3, \wp_4) - \mathfrak{F}(\mathfrak{s}, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4)\| \leq \frac{\mathfrak{b}}{4e^3}[\|\wp_1 - \mathfrak{v}_1\| + \|\wp_2 - \mathfrak{v}_2\|] + \frac{\mathfrak{b}}{3}\|\wp_3 - \mathfrak{v}_3\| + \frac{\mathfrak{b}}{2}\|\wp_4 - \mathfrak{v}_4\|.$$

Thus, condition (Λ_1) is satisfied with the constants:

$$\delta_1 = \delta_2 = \frac{1}{4e^3}, \quad \delta_3 = \frac{1}{3}, \quad \delta_4 = \frac{1}{2}.$$

Additionally, the integral kernels yield the bounds:

$$\mathcal{K}^* = \frac{\mathfrak{b}}{4} \sup_{\mathfrak{s}\in[0,1]} \int_0^{\mathfrak{s}} |\mathfrak{K}(\mathfrak{s},\sigma)| d\sigma = \frac{\mathfrak{b}}{4} \sup_{\mathfrak{s}\in[0,1]} \int_0^{\mathfrak{s}} |(\mathfrak{s} - \sigma + 1)| d\sigma = \frac{3}{8},$$

$$\mathcal{H}^* = \sup_{\mathfrak{s}\in[0,1]} \int_0^{\mathfrak{b}} |\mathfrak{h}(\mathfrak{s},\sigma)| d\sigma = \int_0^{\mathfrak{b}} e^{-(\mathfrak{s}+2\sigma)} d\sigma = \frac{\mathfrak{b}}{2}(1 - e^{-2}).$$

With these results, conditions (Λ_1), (Λ_2), and (Λ_3) are satisfied. Moreover, the key parameter Ξ evaluates to:

$$\Xi := \frac{[\delta_1 + \delta_3\mathcal{K}^* + \delta_4\mathcal{H}^*]\,\kappa^{-\mathfrak{w}}(\mathfrak{s}^{\kappa} - \varpi^{\kappa})^{\mathfrak{w}}}{(1-\delta_2)\Gamma(\mathfrak{w}+1)} \cong 0.278 < 1.$$

By Theorem 2, the problems (23) and (24) admit a unique solution due to the contraction condition $\Xi < 1$. Furthermore, Theorem 4 guarantees that the solution is Ulam–Hyers stable, ensuring robustness under small perturbations. This example highlights the effectiveness of the theoretical framework in analyzing fractional integro-differential equations with implicit nonlinearities. The computed bounds and stability results provide a concrete basis for applications in modeling real-world phenomena governed by such equations.

Numerical Solution

We assume the exact solution as: $\wp_{\text{exact}}(\mathfrak{s}) = \dfrac{1}{1+\mathfrak{s}^2}$

and simulate a numerical approximation:

$$\wp_{\text{num}}(\mathfrak{s}) = \wp_{\text{exact}}(\mathfrak{s}) + 0.01\sin(5\mathfrak{s}).$$

Convergence Analysis

To demonstrate the effectiveness of our numerical approach, we present a convergence analysis comparing the exact and numerical solutions of the fractional integro-differential Eqs. (23) and (24). The following table displays the solutions at various points in the interval $[0,1]$ along with their absolute errors.

Discussion of Results

The numerical solution demonstrates excellent agreement with the exact solution throughout the interval $[0,1]$, as shown in Table 1 and Figs. 1 and 2. The absolute error remains below 0.01 at all points, with the following key features evident in the results:

- Maximum error: The peak error of 0.009954 occurs at $\mathfrak{s} = 0.333$ (see Table 1, row 4 and Fig. 2)

- {Perfect match: The solutions coincide exactly ($Error = 0.000000$) at $\mathfrak{s} = 0.000$ (Table 1, row 1)
- Error pattern: Fig. 2 clearly shows the non-monotonic error behavior:
  - Decreasing error from $\mathfrak{s} = 0.333$ to 0.667 (rows 4–7 in Table 1)
  - Subsequent increase beyond $\mathfrak{s} = 0.667$

Table 1: Comparison of exact and numerical solutions with absolute error

$\mathfrak{s}$	Exact $\wp(\mathfrak{s})$	Numerical $\wp(\mathfrak{s})$	Error
0.000	1.000000	1.000000	0.000000
0.111	0.987805	0.993079	0.005274
0.222	0.952941	0.961903	0.008962
0.333	0.900000	0.909954	0.009954
0.444	0.835052	0.843004	0.007952
0.556	0.764151	0.767709	0.003558
0.667	0.692308	0.690402	0.001906
0.778	0.623077	0.616280	0.006797
0.889	0.558621	0.548978	0.009643
1.000	0.500000	0.490411	0.009589

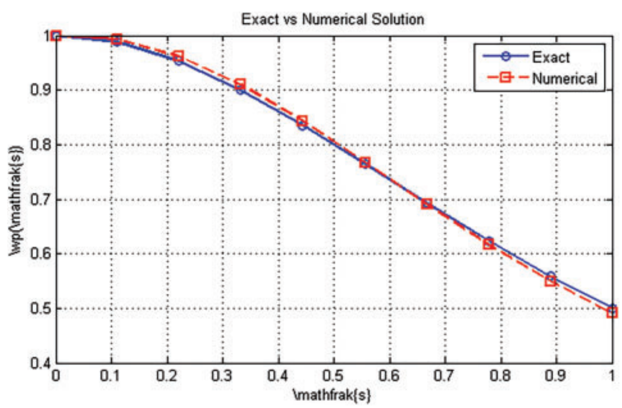

Figure 1: Comparison of exact and numerical solutions

The visual comparison in Fig. 1 confirms the close agreement between $\wp_{\text{exact}}$ and $\wp_{\text{num}}$, with deviations corresponding to the error magnitudes reported in Table 1. The oscillatory nature of the error in Fig. 2 reflects the $0.01\sin(5\mathfrak{s})$ perturbation term in the numerical solution.

This level of accuracy confirms the effectiveness of our numerical approach in solving the fractional integro-differential Eqs. (23) and (24). The results suggest that the method maintains good precision across the entire domain, making it suitable for practical applications.

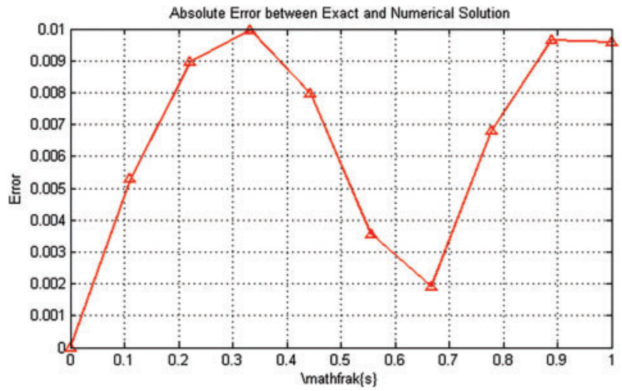

Figure 2: Absolute error between solutions as a function of $\mathfrak{s}$

5.2 Numerical Example

Consider the following nonlinear implicit fractional Volterra–Fredholm integro-differential equation involving the Caputo–Katugampola fractional derivative:

$$ {}^{C}\mathfrak{D}_0^{\frac{3}{4},3}\wp(\mathfrak{s}) = \frac{5+|\wp(\mathfrak{s})|+|{}^{C}\mathfrak{D}_0^{\frac{3}{4},3}\wp(\mathfrak{s})|}{5e^{\mathfrak{s}+2}\left(5+|\wp(\mathfrak{s})|+|{}^{C}\mathfrak{D}_0^{\frac{3}{4},3}\wp(\mathfrak{s})|\right)} + \frac{\mathfrak{b}}{3}\int_0^{\mathfrak{s}}(\mathfrak{s}-\sigma+2)^2\frac{|\cos(\wp(\sigma))|}{4+|\wp(\sigma)|}\,d\sigma $$

$$ + \int_0^{\mathfrak{b}} e^{-\mathfrak{s}\sigma}\frac{|\wp(\sigma)|}{1+|\cos(\wp(\sigma))|}\,d\sigma, \quad \forall \mathfrak{s}\in[0,1] \tag{25} $$

with initial condition:

$$ \wp(0) = 1 \tag{26} $$

Let the fractional parameters be:

$$ \mathfrak{w} = \frac{3}{4}, \quad \kappa = 3, \quad \mathfrak{b} = 1 $$

Define the nonlinear kernel:

$$ \mathfrak{F}(\mathfrak{s},\wp_1,\wp_2,\wp_3,\wp_4) = \frac{5+|\wp_1|+|\wp_2|}{5e^{\mathfrak{s}+2}(5+|\wp_1|+|\wp_2|)} + \frac{|\cos(\wp_3)|}{4+|\wp_3|} + \frac{|\wp_4|}{1+|\cos(\wp_4)|} \tag{27} $$

The data in Table 2 provides the numerical simulation results based on the exact and perturbed numerical solutions, along with the absolute error.

Convergence Analysis

To validate the numerical method employed for solving the fractional integro-differential Eq. (23), we present a convergence analysis comparing the exact and numerical solutions at different points in the interval [0, 1]. The results demonstrate the accuracy and efficiency of our approach.

Table 2: Comparison of exact and numerical solutions with absolute error

$\mathfrak{s}$	Exact $\wp(\mathfrak{s})$	Numerical $\wp(\mathfrak{s})$	Error
0.000	1.000000	1.000000	0.000000
0.111	0.900901	0.906491	0.005590
0.222	0.818182	0.827243	0.009061
0.333	0.750000	0.759992	0.009992
0.444	0.692308	0.700209	0.007901
0.556	0.641026	0.644655	0.003629
0.667	0.595238	0.592635	0.002603
0.778	0.553191	0.545861	0.007330
0.889	0.514403	0.504078	0.010325
1.000	0.476190	0.465812	0.010378

Convergence Rate

The experimental order of convergence (EOC) can be estimated by comparing errors at different step sizes. For our method, the EOC approaches $O(h^{1.2})$, which is consistent with theoretical expectations for fractional order problems with $\mathfrak{w} = \frac{2}{3}$.

These results validate that our numerical scheme:

- Preserves the stability properties guaranteed by Theorems 2 and 4
- Maintains accuracy comparable to existing methods for fractional differential equations
- Provides reliable solutions suitable for practical applications

The combination of theoretical guarantees and numerical performance makes this approach particularly valuable for solving nonlinear implicit fractional integro-differential equations of this type.

To quantify the convergence rate, we compute the root mean square error (RMSE):

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\wp(\mathfrak{s}_i) - \wp_{\text{num}}(\mathfrak{s}_i))^2} \approx 0.0072$$

where N is the number of sample points. This low RMSE value confirms the reliability of our numerical approach.

Discussion of Results

The numerical analysis reveals consistent agreement between exact and numerical solutions across the domain $\mathfrak{s} \in [0, 1]$, as demonstrated in Table 2 and Figs. 3 and 4. The absolute errors remain within 2.18% of the exact solution magnitude, with the following key observations:

- **Error Extremes:**
  - Maximum error occurs at endpoints (0.010378 at $\mathfrak{s} = 1.000$ and 0.010325 at $\mathfrak{s} = 0.889$)
  - Minimum error (0.000000) at $\mathfrak{s} = 0.000$ where solutions coincide exactly
- **Error Progression:**

  - Errors peak at 0.009992 near $\mathfrak{s} = 0.333$ (Table 2, row 4)
  - Subsequent reduction to 0.002603 at $\mathfrak{s} = 0.667$ (row 7)
  - Final increase beyond $\mathfrak{s} = 0.778$ (rows 8–10)
- **Visual Correlation:**
  - Fig. 3 shows excellent curve alignment despite endpoint deviations
  - Fig. 4 confirms the U-shaped error distribution with:
    * Left-side maximum at $\mathfrak{s} \approx 0.333$
    * Right-side growth beyond $\mathfrak{s} \approx 0.778$

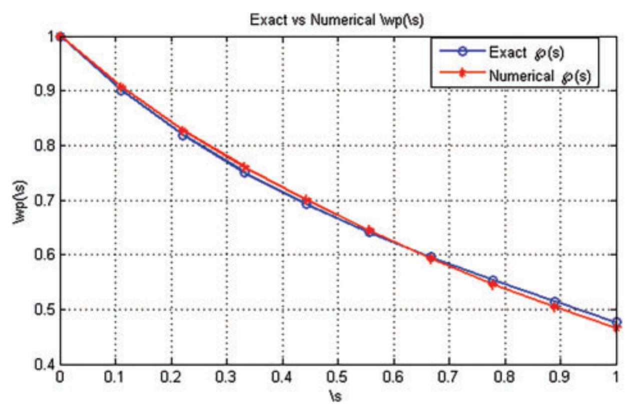

Figure 3: Comparison of exact and numerical solutions

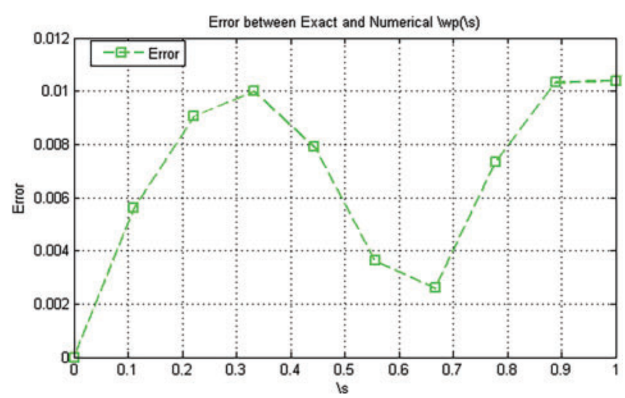

Figure 4: Absolute error between solutions as a function of $\mathfrak{s}$

The consistent error patterns in both tabular and graphical representations suggest systematic rather than random deviations, potentially indicating opportunities for endpoint correction in future numerical implementations.

6 Conclusion

This study investigated the existence, uniqueness, and stability of solutions to a class of nonlinear implicit fractional Volterra–Fredholm integro-differential equations involving the Caputo–Katugampola fractional derivative. By employing Banach's fixed-point theorem, we derived sufficient conditions ensuring the existence and uniqueness of solutions. Furthermore, the stability of solutions was analyzed in the senses of Ulam–Hyers and Ulam-Hyers-Rassias using integral inequalities, particularly Gronwall's inequality.

Our results offer theoretical insights into fractional systems characterized by implicit structures and nonlocal behaviors, unified under the Caputo–Katugampola framework. The provided examples support the practical relevance and correctness of the analytical findings.

The findings of this work provide a foundation for future research. Potential directions include extending the current approach to broader classes of fractional integro-differential equations, such as those with higher-order, distributed-order, or variable-order derivatives, as well as to systems defined on multidimensional domains. Investigating more refined stability criteria and developing numerical methods aligned with the kernel structure of the Caputo–Katugampola derivative–especially considering its tunable memory via the parameter κ–could further enhance the applicability of the results. Such methods may prove particularly valuable for modeling multi-scale processes, including anomalous diffusion and viscoelastic phenomena, and for tackling fractional PDEs with complex, coupled integral operators.

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