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DR-PDEE-BASED EFFICIENT STOCHASTIC DYNAMICAL RESPONSE ANALYSIS FOR HIGH-DIMENSIONAL NONLINEAR SYSTEMS SUBJECT TO MULTIPLICATIVE NON-WHITE EXCITATION

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Abstract. Multiplicative stochastic excitation plays an indispensable role in modeling complex nonlinear dynamical systems across multiple fields of science and engineering. In particular, random noise in physical systems is typically not ideal white noise. The coupling between excitation and system responses in multiplicative problems, combined with the non-white nature of the excitation, leads to significant challenges for probabilistic response analysis, especially in high-dimensional dynamical systems. To address these challenges, an efficient method based on the dimension-reduced probability density evolution equation (DR-PDEE) is proposed for analyzing the response probability density of such systems. In this approach, the multiplicative non-white noise excitation is simulated by a linear analog filter, and the augmented system, which combines the original dynamical system with the linear filter, becomes a higher-dimensional system under additive white noise. Subsequently, by combining the quantity of interest with one filter output that possesses a nonzero intrinsic diffusion function, and solving the two-dimensional DR-PDEE for the joint process, the probability density function (PDF) solution of the quantity of interest can be directly obtained. Notably, due to the introduction of the analog filter, the corresponding intrinsic diffusion function in the DR-PDEE for the joint process becomes a known constant. Therefore, in this method, only the intrinsic drift functions of the DR-PDEE need to be determined, which can be reliably estimated through a small number of deterministic analyses. A numerical example is illustrated to validate the proposed method, demonstrating that the DR-PDEE achieves PDF solutions with high tail accuracy by only a small number of deterministic analyses.

1 INTRODUCTION

Multiplicative noise excitation is frequently encountered in scientific and engineering scenarios and leads to complex nonlinear dynamics [1, 2]. Unlike idealized white noise, random excitations in real-world physical systems typically exhibit non-white characteristics. In science, the correlation time of non-white noise has been found to significantly influence the noise-induced phase transition of nonequilibrium systems [1, 3]. In engineering applications, such as parametric ship rolling in random waves and wind-induced oscillations, the associated excitations typically deviate from white noise assumptions and conform to corresponding power spectral density models [4, 5]. These characteristics underscore the necessity of effective analysis methods for systems under non-white multiplicative noise excitations.

Various approaches have been proposed to analyze the response PDFs of systems under non-white multiplicative noise. The Fokker-Planck-Kolmogorov (FPK) equation method can be applied to such systems by simulating the non-white excitation with an analog filter. However, for high-dimensional nonlinear systems, the corresponding FPK equation becomes a high-dimensional partial differential equation (PDE), which is highly challenging to solve, whether through analytical [6, 7] or numerical methods [8, 9, 10, 11]. In particular, augmenting the original system with a linear filter further increases the dimensionality of the associated FPK equation, compounding the difficulty of solving the PDE. Another frequently used method is the stochastic averaging method [12, 13], but it is restricted to lightly damped dynamical systems under wide-banded stochastic excitations. Some widely used approximation methods, such as the stochastic linearization [14] and the moment closure technique [15], are often combined with additional filtering systems to solve the response statistical moments of the system. However, these methods lack accuracy assurance for complex nonlinear systems like multiplicative excitation problems [9].

Although significant progress has been made, classic random vibration methods for probabilistic response analysis of high-dimensional systems under non-white multiplicative noise still face challenges. In this regard, the recently developed dimension-reduced probability density evolution equation (DR-PDEE) shows promise in circumventing these challenges. In fact, the concept of dimensionality reduction for FPK equations can be traced back to the study of certain special systems in statistical physics [1]. In engineering, Er [16], Chen and Yuan [17], and Jiang et al. [18] developed the formal dimension-reduced FPK equation. Later, Lyu and Chen [19] proposed the DR-PDEE for general path-continuous stochastic processes, which has broader applicability compared to the reduced-dimensional FPK equation. This includes applications to non-Markov systems [20, 21], systems with random parameters [22], and systems excited by anticipating processes. Recently, the DR-PDEE-based method has been extended to solve probabilistic responses of systems under multiplicative white noise excitation, where in the DR-PDEE the intrinsic diffusion function is no longer a known constant [23]. Actually, the responses of systems excited by multiplicative non-white noise also satisfy the path-continuous condition, which promises the extension of the DR-PDEE to multiplicative non-white noise problems.

In this paper, an efficient probabilistic response analysis method for high-dimensional nonlinear systems subjected to multiplicative non-white excitation is presented. In Section 2, the DR-DPEE for system responses under multiplicative non-white noise excitation is constructed by introducing a linear filter to model the non-white noise. The numerical implementation procedure is outlined in Section 3. In Section 4, a typical example is provided to demonstrate the effectiveness of the proposed method. Finally, conclusions are drawn in Section 5.

2 DR-PDEE for high-dimensional nonlinear systems subjected to multiplicative colored noise

Without loss of generality, consider an n-degree-of-freedom dynamical system subjected to multiplicative stochastic excitation, whose equation of motion reads

$$\boldsymbol{M}\ddot{\boldsymbol{X}}(t)+\boldsymbol{f}\left(\boldsymbol{X}(t),\dot{\boldsymbol{X}}(t),t\right)=\boldsymbol{L}\left(\boldsymbol{X}(t),\dot{\boldsymbol{X}}(t),t\right)\boldsymbol{\xi}(t), \tag{1}$$

where $\boldsymbol{X}(t),\dot{\boldsymbol{X}}(t),\ddot{\boldsymbol{X}}(t)\in\mathbb{R}^n$ are the vectors of displacement, velocity, and acceleration, respectively; $\boldsymbol{M}\in\mathbb{R}^{n\times n}$ denotes the mass matrix; $\boldsymbol{f}\left(\boldsymbol{X}(t),\dot{\boldsymbol{X}}(t),t\right)\in\mathbb{R}^n$ is the nonlinear force vector; $\boldsymbol{L}\left(\boldsymbol{X}(t),\dot{\boldsymbol{X}}(t),t\right)\in\mathbb{R}^{n\times r}$ denotes the input location matrix; $\boldsymbol{\xi}(t)\in\mathbb{R}^r$ is the stochastic excitation. If the input location matrix $\boldsymbol{L}\left(\boldsymbol{X}(t),\dot{\boldsymbol{X}}(t),t\right)$ is independent of $\boldsymbol{X}(t)$ and $\dot{\boldsymbol{X}}(t)$, the excitation is referred to as the additive excitation. Otherwise, it is classified as multiplicative [24]. In this paper, the focus is placed on multiplicative stochastic excitation problems.

Consider the excitation $\boldsymbol{\xi}(t)$ as non-white noise with a bounded mean square value. Then, by denoting $\boldsymbol{V}(t)=\dot{\boldsymbol{X}}(t)$ and introducing the state variable $\boldsymbol{Z}(t)=\left(\boldsymbol{X}^\top(t),\boldsymbol{V}^\top(t)\right)^\top$, the corresponding state function of Eq. (1) is

$$\mathrm{d}\boldsymbol{Z}(t)=\boldsymbol{a}(\boldsymbol{Z}(t),t)\,\mathrm{d}t+\boldsymbol{B}(\boldsymbol{Z}(t),t)\,\boldsymbol{\xi}(t)\,\mathrm{d}t \tag{2}$$

with

$$\boldsymbol{a}(\boldsymbol{Z}(t),t)=\begin{pmatrix}\boldsymbol{V}(t)\\-\boldsymbol{M}^{-1}\boldsymbol{f}(\boldsymbol{X}(t),\boldsymbol{V}(t),t)\end{pmatrix},\ \boldsymbol{B}(\boldsymbol{Z}(t),t)=\begin{bmatrix}\boldsymbol{0}_{n\times r}\\\boldsymbol{M}^{-1}\boldsymbol{L}(\boldsymbol{X}(t),\boldsymbol{V}(t),t)\end{bmatrix}. \tag{3}$$

Note that since the excitation $\boldsymbol{\xi}(t)$ has a bouned mean square value, different interpretations of the integral with respect to $\boldsymbol{\xi}(t)$ do not change the form of the state equation, and thus the Wong–Zakai correction term equals zero [24, 19, 25].

Usually, when modeling the system dynamics, the random excitation $\boldsymbol{\xi}(t)$ is assumed to satisfy certain statistical characteristics. For example, for stationary excitation, its mean and power spectral density (PSD) function are typically specified. In such cases, if the PSD function of the excitation can be well approximated by an exponential function or a rational function, the excitation can be viewed as a linear combination of the outputs of an s-dimensional linear filter with white noise as its input, which is

$$\begin{cases}\boldsymbol{\xi}(t)=\boldsymbol{P}\boldsymbol{U}(t)\\\mathrm{d}\boldsymbol{U}(t)=\boldsymbol{K}^{\mathrm{f}}\boldsymbol{U}(t)\,\mathrm{d}t+\boldsymbol{B}^{\mathrm{f}}\mathrm{d}\boldsymbol{W}(t)\,.\end{cases} \tag{4}$$

In Eq. (4), $\boldsymbol{P}\in\mathbb{R}^{r\times s}$ is the linear combination coefficient matrix; $\boldsymbol{K}^{\mathrm{f}}\in\mathbb{R}^{s\times s}$ and $\boldsymbol{B}^{\mathrm{f}}\in\mathbb{R}^{s\times r_0}$ are the parameters of the linear filter; and $\boldsymbol{W}(t)\in\mathbb{R}^{r_0}$ is a Wiener process vector with the covariance matrix of its increment given by $\mathcal{E}\left[\Delta\boldsymbol{W}(t)\,\Delta\boldsymbol{W}^\top(t)\right]=\boldsymbol{D}\Delta t$. Here, $\mathcal{E}(\cdot)$ denotes the mathematical expectation operator. The parameters $\boldsymbol{P}$, $\boldsymbol{K}^{\mathrm{f}}$, $\boldsymbol{B}^{\mathrm{f}}$, and $\boldsymbol{D}$ are carefully designed to ensure that $\boldsymbol{\xi}(t)$ simulated by Eq. (4) satisfies the prescribed PSD function [26, 27, 28].

By combining the original system [Eq. (2)] with the filter [Eq. (4)], an augmented system is obtained, represented by the following Itô SDE:

$$\begin{pmatrix}\mathrm{d}\boldsymbol{Z}(t)\\\mathrm{d}\boldsymbol{U}(t)\end{pmatrix}=\begin{pmatrix}\boldsymbol{a}(\boldsymbol{Z}(t),t)+\boldsymbol{B}\boldsymbol{P}\boldsymbol{U}(t)\\\boldsymbol{K}^{\mathrm{f}}\boldsymbol{U}(t)\end{pmatrix}\mathrm{d}t+\begin{bmatrix}\boldsymbol{0}_{2n\times 1}\\\boldsymbol{B}^{\mathrm{f}}\end{bmatrix}\mathrm{d}\boldsymbol{W}(t) \tag{5}$$

As shown in Eq. (5), by constructing a linear filter to simulate the non-white excitation and combining it with the original system, the multiplicative non-white noise problem is converted into an additive white noise problem. In the augmented system, the coupling between the state variable and the excitation becomes a nonlinear term in the drift function, which more clearly reveals the nonlinear effects of multiplicative noise excitation.

The system response of the augmented system is governed by a $(2n+s)$-dimensional FPK equation. When the equation of motion [Eq. (1)] involves a large number of degrees of freedom, it becomes very difficult to obtain the response PDF by directly analyticlally or numerically solving the corresponding FPK equation [6, 7, 8, 9, 10, 11]. In particular, the linear filter exacerbates the dimensionality issue of the FPK equation.Nevertheless, no matter how many dimensions are involved in the augmented system, the DR-PDEE is a low-dimensional PDE that can be applied to efficiently obtain the response PDF. For the response component of interest, say $Z_\ell(t)$, the ℓ-th component of $\boldsymbol{Z}(t)$, the DR-PDEE governing its instantaneous PDF is a one-dimensional PDE, where the intrinsic drift and diffusion functions are

$$\begin{cases} a_{Z_\ell}^{\text{int}}(z_\ell,t) = \lim\limits_{\Delta t\to 0}\dfrac{1}{\Delta t}\mathcal{E}\left[\Delta Z_\ell(t)|\, Z_\ell(t)=z_\ell\right] = \mathcal{E}\left[a_\ell(\boldsymbol{Z}(t)) + [\boldsymbol{BPU}(t)]_\ell|\, Z_\ell(t)=z_\ell\right], \\ b_{Z_\ell}^{\text{int}}(z_\ell,t) = \lim\limits_{\Delta t\to 0}\dfrac{1}{\Delta t}\mathcal{E}\left\{[\Delta Z_\ell(t)]^2\Big|\, Z_\ell(t)=z_\ell\right\} = 0, \end{cases} \tag{6}$$

respectively. Note that the intrinsic diffusion function of $Z_\ell(t)$ becomes zero, and the corresponding DR-PDEE is reduced to a Liouville-like PDE, i.e.,

$$\frac{\partial p_{Z_\ell}(z_\ell,t)(t)}{\partial t} = -\frac{\partial a_{Z_\ell}^{\text{int}}(z_\ell,t)\, p_{Z_\ell}(z_\ell,t)}{\partial z_\ell}. \tag{7}$$

To avoid the inconvenience of numerically solving the DR-PDEE with a zero intrinsic diffusion function, the quantity of interest, $Z_\ell(t)$, can be solved combined with another process that has a non-zero intrinsic diffusion function. Since the linear filter in Eq. (4) is excited by white noise, there exists at least one component of $\boldsymbol{U}(t)$, say $U_\kappa(t)$ with $\kappa \in \{1,2,\cdots,s\}$, that has a non-zero intrinsic diffusion function. Consequently, the instantaneous PDF of the joint process $(Z_\ell(t), U_\kappa(t))^\top$ is governed by the DR-PDEE [19]

$$\begin{aligned} \frac{\partial p_{Z_\ell U_\kappa}(z_\ell,u_k,t)}{\partial t} = &-\frac{\partial a_{Z_\ell U_\kappa,1}^{\text{int}}(z_\ell,u_\kappa,t)\, p_{Z_\ell U_\kappa}(z_\ell,u_k,t)}{\partial z_\ell} - \frac{\partial a_{Z_\ell U_\kappa,2}^{\text{int}}(z_\ell,u_\kappa,t)\, p_{Z_\ell U_\kappa}(z_\ell,u_k,t)}{\partial u_\kappa} \\ &+ \frac{1}{2}\frac{\partial^2 b_{Z_\ell U_\kappa,22}^{\text{int}}(z_\ell,u_\kappa,t)\, p_{Z_\ell U_\kappa}(z_\ell,u_k,t)}{\partial u_\kappa^2}, \end{aligned} \tag{8}$$

where the intrinsic drift and diffusion fucntions of $(Z_\ell(t), U_\kappa(t))^\top$ are, respectively, [19]

$$\begin{cases} a_{Z_\ell U_\kappa,1}^{\text{int}}(z_\ell,u_\kappa,t) = \lim\limits_{\Delta t\to 0}\dfrac{1}{\Delta t}\mathcal{E}\left[\Delta Z_\ell(t)|\, Z_\ell(t)=z_\ell, U_\kappa(t)=u_\kappa\right] \\ \qquad\qquad = \mathcal{E}\left\{a_\ell(\boldsymbol{Z}(t)) + [\boldsymbol{BPU}(t)]_\ell|\, Z_\ell(t)=z_\ell, U_\kappa(t)=u_\kappa\right\} \\ a_{Z_\ell U_\kappa,2}^{\text{int}}(z_\ell,u_\kappa,t) = \lim\limits_{\Delta t\to 0}\dfrac{1}{\Delta t}\mathcal{E}\left[\Delta U_\kappa(t)|\, Z_\ell(t)=z_\ell, U_\kappa(t)=u_\kappa\right] \\ \qquad\qquad = \mathcal{E}\left\{\left[\boldsymbol{K}^{\text{f}}\boldsymbol{U}(t)\right]_\kappa\Big|\, Z_\ell(t)=z_\ell, U_\kappa(t)=u_\kappa\right\} \\ b_{Z_\ell U_\kappa,22}^{\text{int}}(z_\ell,u_\kappa,t) = \lim\limits_{\Delta t\to 0}\dfrac{1}{\Delta t}\mathcal{E}\left\{[\Delta U_\kappa(t)]^2\Big|\, Z_\ell(t)=z_\ell, U_\kappa(t)=u_\kappa\right\} = \left[\boldsymbol{B}^{\text{f}}\boldsymbol{D}\left(\boldsymbol{B}^{\text{f}}\right)^\top\right]_{\kappa\kappa} > 0. \end{cases} \tag{9}$$

It is evident from Eq. (9) that the intrinsic diffusion function of $(Z_\ell(t), U_\kappa(t))^\top$ is a known constant for multiplicative non-white noise problems, which is different from that in the multiplicative white noise excitation cases [23].

After determining the intrinsic drift functions $a^{\text{int}}_{Z_\ell U_\kappa,1}(z_\ell, u_\kappa, t)$ and $a^{\text{int}}_{Z_\ell U_\kappa,2}(z_\ell, u_\kappa, t)$, numerically solving the two-dimensional DR-PDEE [Eq. (8)] yields the instantaneous PDF of $(Z_\ell(t), U_\kappa(t))^\top$. The PDF of $Z_\ell(t)$ can then be easily obtained as

$$p_{Z_\ell}(z_\ell, t) = \int_{-\infty}^{\infty} p_{Z_\ell U_\kappa}(z_\ell, u_k, t)\,\mathrm{d}u_\kappa. \tag{10}$$

3 Numerical procedure

Solving the probabilistic response of high-dimensional dynamical systems subjected to multiplicative non-white noise generally involves the following three steps:

(1) Constructing a linear filter to simulate non-white excitation $\boldsymbol{\xi}(t)$. This step depends on the choice of the stochastic excitation model. Many classical stochastic excitation models commonly used in engineering can be simulated by linear systems driven by white noise. Typical examples include random processes with exponential autocorrelation functions, which can be modeled by Ornstein–Uhlenbeck (OU) processes [24]. The Kanai–Tajimi [29, 30] and Clough–Penzien [31] spectra for ground acceleration processes can be modeled by SDOF and 2-DOF systems, respectively. Regarding ocean waves, Luo et al. [27] and Chen et al. [28] proposed filtering equations to simulate random waves conforming to the JONSWAP spectrum [4]. In particular, the filter proposed by Chen et al. [28] endows the parameters of the filtering equations with physical meanings, thereby avoiding the need to re-identify the filtering parameters under different wave load conditions. For the stochastic fluctuating wind speed process, the filtering equations can be found in Refs. [32] and [33].

(2) Estimating the intrinsic drift functions of the joint process $(Z_\ell(t), U_\kappa(t))^\top$. As shown in Eq. (9), the intrinsic drift functions of $(Z_\ell(t), U_\kappa(t))^\top$ at a particular instant are essentially the conditional expectations of the system responses at that time. Exact intrinsic drift functions are available for some special systems, such as linear systems [34, 25] and equipartition systems [35] under additive white noise excitation. For most general nonlinear systems, the intrinsic drift functions can be numerically estimated from data obtained through deterministic analyses. Various numerical estimators of the intrinsic drift function have been proposed, including the locally weighted scatter plot smoothing (LOWESS) [36, 19], the copula function method [22], the tailed enhanced method [37], and neural networks [38, 39].

(3) Numerically solving the DR-PDEE [Eq. (8)] to obtain the transient PDF of $(Z_\ell(t), U_\kappa(t))^\top$ and then the PDF of $Z_\ell(t)$ by Eq. (10). As the DR-PDEE is a two-dimensional FPK-like PDE, various numerical technique proposed for solving the FPK equation can be applied to solve the DR-PDEE. In this study, the path integral solution (PIS) method proposed by Yu et al. [40] is adopted. Alternatives include the finite different method [8], the finite element method [9], and recently developed machine learning-based solvers [41, 42, 38].

4 Numerical Example

In this section, a multi-dimensional OU system under multiplicative non-white noise is employed to illustrate the effectiveness of the proposed method. The system is governed by the following m-

dimensional state equation:

$$\mathrm{d}\boldsymbol{Z}(t) = -\boldsymbol{A}\boldsymbol{Z}(t)\,\mathrm{d}t + (\boldsymbol{G}\boldsymbol{Z}(t) + \boldsymbol{G})\,\xi(t), \tag{11}$$

where $\boldsymbol{Z}(t) = (Z_1(t), Z_2(t), \cdots, Z_m(t))^\top \in \mathbb{R}^m$ and $m = 8$ for this example. In the equation, the drift and diffusion coefficients are both linear functions of state variable $\boldsymbol{Z}(t)$ with the corresponding linear coefficients taken as

$$\boldsymbol{A} = \begin{bmatrix} 0.3 & -0.04 & & & & 0 \\ -0.04 & 0.2 & -0.04 & & & \\ & -0.04 & 0.2 & -0.04 & & \\ & & & \ddots & & \\ 0 & & & & -0.04 & 0.1 \end{bmatrix}, \quad \boldsymbol{G} = g\boldsymbol{E}_{m\times m}, \quad \boldsymbol{G}_0 = g_0\mathbf{1}_{m\times 1}. \tag{12}$$

In Eq. (12), $\boldsymbol{E}_{m\times m}$ denotes the the identity diagonal matrix of order m, and $\mathbf{1}_{m\times 1}$ denote an m-dimensional column vector whose components are all ones. The initial condition of the system is $\boldsymbol{Z}(t_0) = \mathbf{0}_{m\times 1}$.

The non-white noise $\xi(t)$ is considered a zero-mean stationary Gaussian process with an exponential-type autocorrelation function, which is given by:

$$\mathcal{E}[\xi(t)\,\xi(t+\tau)] = \frac{D_0}{2k}e^{-k|\tau|}, \tag{13}$$

where $k = 0.1$ and $D_0 = 0.02$ in this example. The Gaussian excitation with the autocorrelation function given in Eq. (13) can be simulated by the following linear filter [24]

$$\mathrm{d}\xi(t) = -k\xi(t)\,\mathrm{d}t + \mathrm{d}W(t) \tag{14}$$

where $W(t)$ is a one-dimensional Wiener process whose increments satisfy $\mathcal{E}\left\{[\Delta W(t)]^2\right\} = D_0\Delta t$. The initial condition of the linear filter is $\xi(t_0) \sim \mathcal{E}(0, D_0/(2k))$.

After constructing the linear filter to simulate the colored noise excitation, combining it with the original state equation [Eq. (11)] yields an augmented system:

$$\begin{pmatrix} \mathrm{d}\boldsymbol{Z}(t) \\ \mathrm{d}\xi(t) \end{pmatrix} = \begin{pmatrix} -\boldsymbol{A}\boldsymbol{Z}(t) + (\boldsymbol{G}\boldsymbol{Z}(t) + \boldsymbol{G}_0)\,\xi(t) \\ -k\xi(t) \end{pmatrix}\mathrm{d}t + \begin{bmatrix} \mathbf{0}_{m\times 1} \\ 1 \end{bmatrix}\mathrm{d}W(t), \tag{15}$$

which is driven by additive white noise. It is evident from the augmented system that the coupling between $\boldsymbol{Z}(t)$ and $\xi(t)$ forms the nonlinear term of the drift coefficient for the augmented system.

Suppose the response process $Z_\ell(t)$ is of interest. The PDF of $Z_\ell(t)$ can be solved by combing $Z_\ell(t)$ with $\xi(t)$, whose instantaneous joint PDF, denoted as $p_{Z_\ell\xi}(z_\ell, u, t)$, is governed by the following DR-DPEE:

$$\begin{aligned} \frac{\partial p_{Z_\ell\xi}(z_\ell, u, t)}{\partial t} = & -\frac{\partial a^{\mathrm{int}}_{Z_\ell\xi,1}(z_\ell, u, t)\,p_{Z_\ell\xi}(z_\ell, u, t)}{\partial z_\ell} - \frac{\partial a^{\mathrm{int}}_{Z_\ell\xi,2}(z_\ell, u, t)\,p_{Z_\ell\xi}(z_\ell, u, t)}{\partial u} \\ & + \frac{1}{2}\frac{\partial^2 b^{\mathrm{int}}_{Z_\ell\xi,22}(z_\ell, u, t)\,p_{Z_\ell\xi}(z_\ell, u, t)}{\partial u^2}, \end{aligned} \tag{16}$$

where the intrinsic drift and diffusion functions are, respectively,

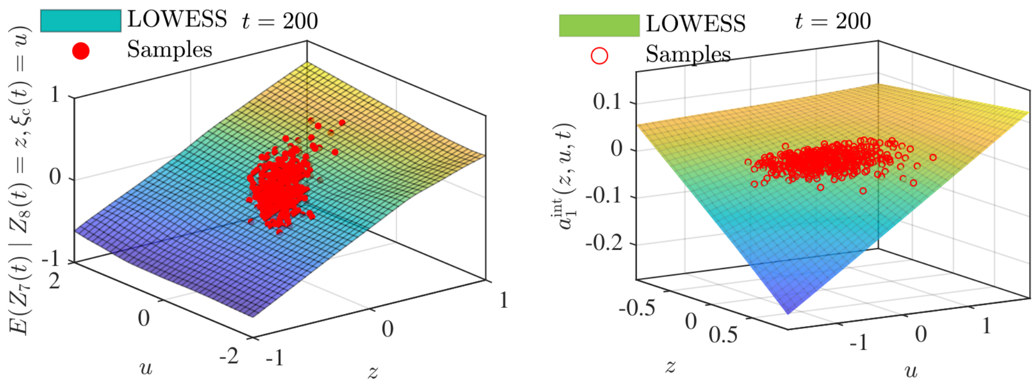
$$\begin{cases} a^{\text{int}}_{Z_\ell\xi,1}(z_\ell,u,t) = \mathcal{E}\left\{\left[-\boldsymbol{A}\boldsymbol{Z}(t) + \left(\boldsymbol{G}\boldsymbol{Z}(t)+\boldsymbol{G}_0\right)\xi(t)\right]_\ell \middle| Z_\ell(t)=z_\ell, \xi(t)=u\right\} \\ \qquad\qquad\qquad = -A_{\ell\ell}z_\ell + (gz_\ell+g_0)\,u - \sum\limits_{j\neq\ell}\mathcal{E}\left[A_{\ell j}Z_j(t)\middle| Z_\ell(t)=z_\ell,\xi(t)=u\right] \\ a^{\text{int}}_{Z_\ell\xi,2}(z_\ell,u,t) = \mathcal{E}\left\{-k\xi(t)\middle| Z_\ell(t)=z_\ell,\xi(t)=u\right\} = -ku \\ b^{\text{int}}_{Z_\ell\xi,22}(z_\ell,u,t) = D_0. \end{cases} \tag{17}$$

As shown in Eq. (17), the intrinsic diffusion function of $\left(Z_\ell(t), \xi(t)\right)^\top$ is a known constant. As for the intrinsic drift functions, part of $a^{\text{int}}_{Z_\ell\xi,1}(z_\ell,u,t)$ and $a^{\text{int}}_{Z_\ell\xi,2}(z_\ell,u,t)$ are determined by taking the form of conditional expectations. Consequently, the unknown terms are the conditional expectations $\mathcal{E}\left\{A_{\ell j}Z_j(t) \mid Z_\ell(t)=z_\ell,\xi(t)=u\right\}$ ($j\neq\ell$), which depend on the correlation among $Z_j(t)$, $Z_\ell(t)$, and $\xi(t)$ and can be determined from the data of the response trajectories.

Here, let $\ell=8$, i.e., the 8-th response component of the state vector is of concern. Thus, the conditional expectation to be determined is $\mathcal{E}\left[Z_7(t)\middle| Z_8(t)=z_\ell,\xi(t)=u\right]$, which is estimated by the LOWESS method [36] with the sample trajectories obtained from 500 deterministic analyses of the augmented system. The fourth stochastic Runge-Kutta scheme [43] is adopted with the time step being 0.005 and the total duration being 200. In estimating the conditional expectation by the LOWESS technique, the ratio of the sample employed for local weighted regression is set as 0.15. The estimated surfaces of the conditional expectation $\mathcal{E}\left[Z_7(t)\middle| Z_8(t)=z_\ell,\xi(t)=u\right]$ and the corresponding intrinsic drift function $a^{\text{int}}_{Z_\ell\xi,1}(z_\ell,u,t)$ at a typical time instant are plotted in Fig.1. It is evident from Fig.1a that the conditional expectation $\mathcal{E}\left[Z_7(t)\middle| Z_8(t)=z_\ell,\xi(t)=u\right]$ is approximately planar, while in Fig. 1b, the intrinsic drift function $a^{\text{int}}_{Z_\ell\xi,1}(z_\ell,u,t)$ exhibits significant nonlinearity, which mainly originates from the multiplicative excitation term that serves as the nonlinear terms of the augmented system.

After determining the intrinsic drift and diffusion functions, the PIS method is employed to solve the two-dimensional DR-PDEE [40], where the time step is set to 2, and the solution domain is set to $[-0.5,1.4]\times[-2,2]$. The solution domain is uniformly divided into 80 subintervals along both the z_8 and u directions, with second-order Gaussian–Legendre quadrature points placed within each subinterval.

The PDF of $Z_8(t)$ obtained by solving the DR-PDEE is compared with the MCS statistics involving 10^6 simulations. The PDFs and cumulative distribution function (CDFs) at representative instants are plotted in Figs.2 - 3. It is observed from Fig.2 that $Z_8(t)$ exhibits significant non-Gaussianity and skewness. Further, the joint PDF of $Z_8(t)$ and $\xi(t)$ at typical instants is shown in Fig.4, revealing a nonlinear dependence between the two processes. This reveals that multiplicative excited systems are essentially nonlinear systems, even though the drift part looks like in a linear form. From the comparison with the MCS results in these figures, the instantaneous PDF of the system response is accurately captured by the DR-PDEE. Notably, as shown in Fig. 3, the predicted system response CDFs, based on only 500 deterministic analyses, show high accuracy even at the magnitude level of $10^{-4}\sim10^{-5}$.



(a) Conditional expectation (b) Intrinsic drift function

Figure 1: Estimated conditional expectation and intrinsic drift function for the multidimensional OU system at typical instants.

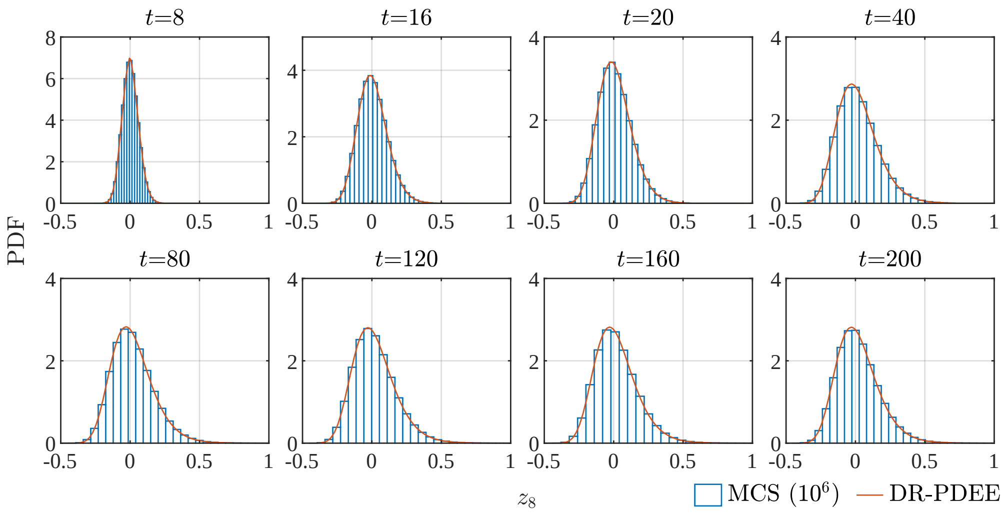


Figure 2: The response PDFs of the multidimensional OU system at typical instants.

5 Conclusions

In the present paper, a new DR-PDEE-based method for response PDF analysis of high-dimensional systems under multiplicative non-white noise is proposed. By introducing an analog filter to simulate the non-white excitation, the augmented system consisting of the original system and the filter is reduced to an additive noise-excited system. Regardless of the dimensionality involved in the augmented system, the response PDF can be directly obtained by solving the DR-PDEE, a two-dimensional PDE governing the joint PDF of the quantity of interest and the filter output, which has a non-zero intrinsic diffusion function. Notably, the intrinsic diffusion function in the DR-PDEE for multiplicative non-white noise problems is constant, which differs from multiplicative white noise excitation problems. An example is provided to illustrate the effectiveness of the proposed method, where high PDF accuracy, around $10^{-4} \sim 10^{-5}$, is achieved with 500 deterministic analyses. Future work will explore stability analysis based on the DR-PDEE.

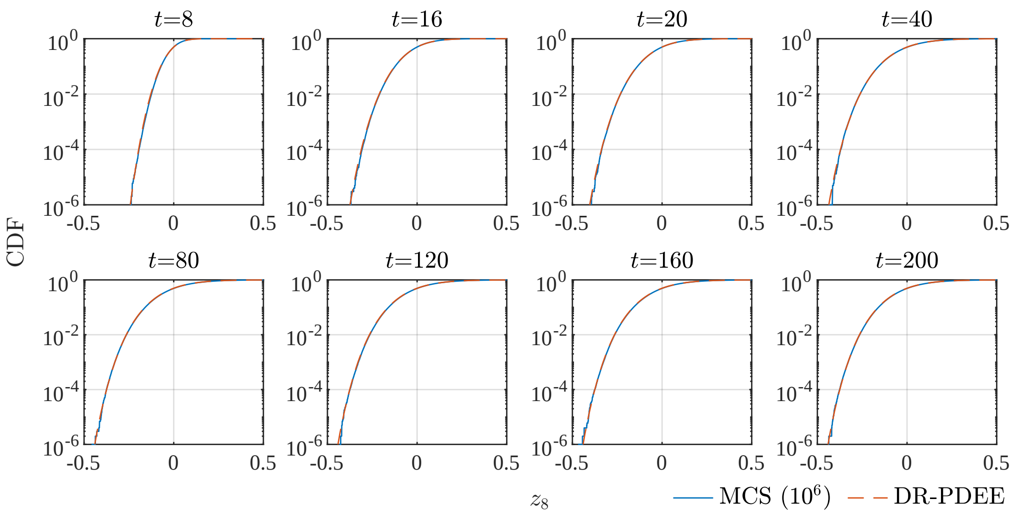


Figure 3: The response CDFs of the multidimensional OU system at typical instants.

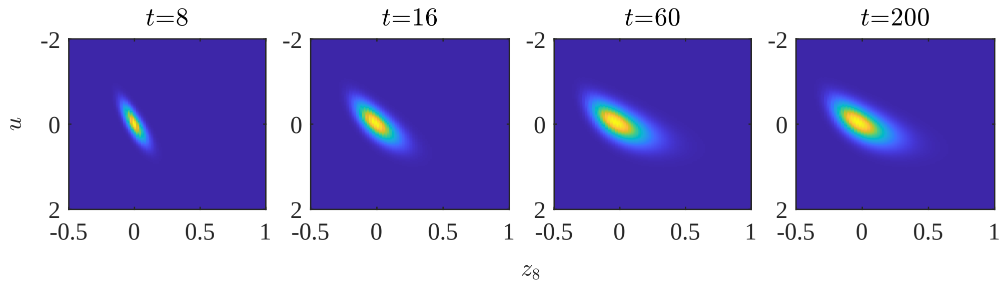


Figure 4: The joint PDFs of $Z_8\,(t)$ and $\xi\,(t)$ for the mulit-dimensional OU system at typical instants.

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